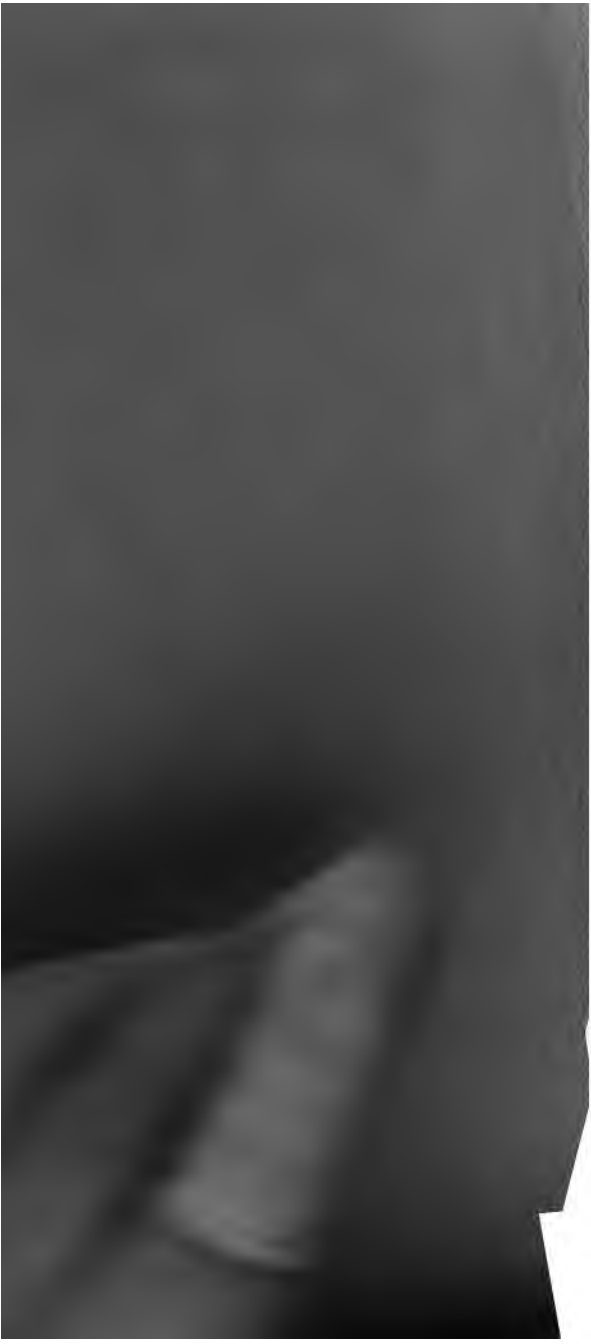

THE

INSTITUTES

OF

ALGEBRA.

BEING

THE FIRST PART OF A COURSE OF MATHEMATICS,

DESIGNED FOR THE

USE OF SCHOOLS, ACADEMIES, AND COLLEGES.

BY

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TO THE

PRINCIPALS OF THE WARD AND PUBLIC SCHOOLS
IN THE CITY OF NEW YORK,

WHOSE EFFORTS IN THE CAUSE OF EDUCATION HAVE RAISED THOSE INSTITUTIONS TO AN EMINENCE ALIKE HONORABLE TO THE OFFICERS, AND A SOURCE OF CONGRATULATION AND PRIDE TO THE PUBLIC,

This Volume is respectfully Dedicated

BY ITS AUTHOR.

New York, August, 1852.

PREFACE.

In presenting this volume to the public, the author begs leave respectfully to remark that his object is to offer a work that would materially lighten the labor of the instructor, and facilitate the progress of the pupil. The experience of twenty-five years in teaching the Mathematics has convinced him that the science of Algebra is the most difficult branch that the student encounters. It is commenced before the mind has had much previous training; the language is new to him, and he requires, therefore, all the light that ingenuity can shed upon it to insure him permanent success. In the following pages the subject is handled in a plain and familiar manner. The spirit of the recitation-room has been embodied, as far as it was possible, in order that the science might be made attractive and interesting to the youthful mind.

If the attainment of a good style in literary compositions is to be reached by following the example of correct and polished writers, the same remark is true in reference to analytical compositions. A good style is always to be sought for; it is always preferable to a loose, obscure, or an inelegant one, let the subject be what it may. How far the author has succeeded in imparting a correct style is left to the candid criticism of impartial judges.

He offers no apology for the appearance of his "Institutes," conscious that no apology is needed. If he has presented the science of Algebra in a way that will promote its study, or advance the interest of scientific education, apology would be unnecessary. If he has not—if his work is found wanting in any essentials—no prefatorial apology would be accepted as an excuse for so grave an error.

He may be permitted to remark further, that this volume embraces all that is necessary for a collegiate course. To give more than this was not his object; still far less was it his intention to fall short of this limit.

There are already many valuable treatises on Algebra, which contain matters essential to the mathematician, but which can not be taught in the time devoted to an undergraduate course of education. A work, therefore, that shall contain the requisites for such a course, elaborately executed, has been demanded, and the present volume is offered to the scientific public as constituting that work.

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ELEMENTS OF ALGEBRA.

DEFINITIONS.

(*Art.* 1.) ALGEBRA may be considered as a language of symbols. Its nature is such that it may be applied to any branch of science, provided the principles of the science to which it is applied are understood. Its great utility consists in the generality of its operations, and the comprehensiveness of its formulas.

(2.) Generally, known quantities are represented by the first letters of the alphabet, as a, b, c, &c., and unknown quantities by the final letters, as x, y, z, &c. The other symbols are,

(3.) $+$ *plus*, signifying addition.

$-$ *minus*, denoting that the quantity to the right is to be subtracted from that on the left of it.

These signs always affect the quantity that follows them, and are to be interpreted in a contrary sense; that is, if *plus* signifies *upward*, *forward*, *to the right*, *gain*, *addition*, time future, &c., then *minus* is to be interpreted *downward*, *backward*, *to the left*, *loss*, subtraction, time past, &c.

(4.) The sign $\times$ is called the sign of multiplication, and when placed between two quantities, denotes that they are to be multiplied together. The multiplication of quantities may also be indicated by placing a point between them.

(5.) When the quantities are represented by letters, the usual mode of indicating their product is to write the letters one after another in alphabetical order, without interposing any sign or point between them.

(6.) In the product of several letters, as abc, each letter is called a *literal factor*.

(7.) There are three signs by which division is designated; these are $a \div b$, $\frac{a}{b}$, and $a\underline{|b}$, each of which denotes that a is to be divided by b.

(8.) The sign of $=$ is read *equal to*, and when placed between two quantities denotes that they are equal to each other; thus, $20-5=15$,
is read *twenty, minus five, is equal to fifteen.*

Also, $a+b=c-d$,
indicates that the sum of a and b is equal to the difference between c and d.

(9.) The sign of inequality is expressed by $>$ or $<$, and is used to denote that one quantity is greater or less than another.

Thus, $9>5$ is read 9 is greater than 5, and $m<n$ is read m is less than n; the opening of the sign being turned toward the greater quantity.

(10.) These dots $\therefore$ are used to denote therefore, and ∞ infinity. Other symbols are employed which will be explained in their proper places.

(11.) If a quantity is added to itself several times, as

$$a+a+a+a,$$

it is usually written but once, and a figure is then placed before it, to show the number of times it is taken; thus,

$$a+a+a+a+a+a=6a.$$

The number 6 is called the coefficient of a, and denotes that a is taken six times.

In the expression $6ax$, the number 6 may be called the coefficient of ax, or the quantity $6a$ may be regarded as the coefficient of x. Therefore,

A *coefficient* is a quantity placed before another quantity, to show the number of times the latter quantity is to be taken. It also denotes one more than the number of times the quantity is added to itself. When no coefficient is written, the coefficient *one* is always understood.

(12.) If a quantity be multiplied by itself a number of times, as $aaaaaa$,
the product is usually expressed by writing the letter once, and placing a number to the right of and a little above it; thus,

$$aaaaaa=a^6.$$

The number 6 is called the exponent of a, and denotes the number of times which a enters as a factor.

Hence the *exponent* of a quantity shows the number of times the quantity is taken as a factor. It also indicates the number of times, *plus one*, that the quantity is to be multiplied by itself. When there is no exponent expressed, the exponent 1 is always understood.

(13.) The product arising from multiplying a quantity by itself any number of times is called the power of that quantity, and the exponent denotes the *degree* of the power. Thus,

$a^1=a$ is the first power of a,
$a^2=aa$ is the second power of a,
$a^3=aaa$ is the third power of a,

in which the exponents of the powers are 1, 2, 3, and the powers themselves are the results of the multiplication.

(14.) The root of a quantity is a quantity which, being multiplied by itself a certain number of times, will produce the given quantity.

$\sqrt{\ }$ is called the radical sign, and when prefixed to a quantity, indicates that its root is to be extracted.

Thus, $\sqrt{a}$ denotes the square root of a,

$\sqrt[3]{a}$ denotes the cube root of a.

The number placed over the radical sign is called the index of the root. These expressions may also be written thus:

$a^{\frac{1}{2}}$ denotes the square root of a,

$a^{\frac{1}{3}}$ denotes the cube root of a,

$a^{\frac{2}{3}}$ denotes the cube root of the square of a.

(15.) The *reciprocal* of a quantity is unity, divided by that quantity. Thus,

$\frac{1}{a}$ is the reciprocal of a,

and $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$.

(16.) When a quantity is written by means of letters and signs, it is called an *algebraic quantity.* Thus,

$5a$ is an algebraic quantity or expression denoting five times the quantity represented by a.

$3a^3b^2$ is an algebraic expression denoting that three times the cube of a is to be multiplied by the square of b.

(17.) A *monomial* or term is a single algebraic expression, as $2a$, $3b$, $10a^2b^3c^4$.

(18.) A *polynomial* is an algebraic expression consisting of two or more terms connected by the signs of *plus* or *minus*, as

$$7a^2-5bc \text{ and } 5ax+3bc+7d.$$

A polynomial composed of two terms is called a *binomial*, and a polynomial of three terms is called a *trinomial.*

(19.) The *numerical value* of a polynomial is the number obtained by giving a particular value to each of the different letters which enter it, and performing the arithmetical operations indicated by the signs. This value is not affected by changing the order of the terms, provided the signs of the terms remain unchanged.

Examples.

1. Find the numerical value of the polynomial

$$a^2+2ab+b^2,$$

when $a=3$, $b=2$. *Ans.* 25.

2. Find the numerical value of the polynomial

$$(a^2+ab+b^2)(a+b-c),$$

when $a=4$, $b=3$, $c=2$. *Ans.* 185.

3. Find the numerical value of the polynomial

$$(ac-ab+bc)(a^2+ab+b^2),$$

when $a=4$, $b=3$, $c=2$. *Ans.* 74.

4. Find the numerical value of the expression

$$\frac{a^2+b^2-c^2}{ab-ac+bc},$$

when $a=4$, $b=3$, $c=2$. *Ans.* $2\frac{1}{10}$.

5. Find the numerical value of the expression

$$\frac{a+b}{a-c}+\frac{a-c}{b+c}-\frac{a-b}{a+b},$$

when $a=4$, $b=3$, $c=2$. *Ans.* $3\frac{53}{70}$.

(20.) The terms of a polynomial which are preceded by the sign + are called *additive terms*, and those preceded by the sign − are called *subtractive terms.*

When the first term of a polynomial is plus, the sign is generally omitted; and when no sign is expressed, plus is always understood.

(21.) Each literal factor which enters a term is called a *dimension*, and the number of dimensions constitutes the *degree* of the term. Thus,

$7a$ is a term of one dimension, or of the first degree.

$7ab$ is a term of two dimensions, or of the second degree.

(22.) *The degree of a term is determined by taking the sum of the exponents of the letters which enter the term.* Thus,

$5a^3b^2c^2d$ is of the eighth degree, since $3+2+2+1=8$.

(23.) A polynomial is said to be *homogeneous* when all its terms are of the same degree. Thus,

$5a-2b+3c$ is a polynomial of the first degree, and homogeneous.

$7a^2+3bc-5b^2+6c^2$ is a polynomial of the second degree, and homogeneous.

$7a^2+3bc-5b+6c$ is not homogeneous.

(24.) A *vinculum* ——, or a *parenthesis* ( ), is used to express that all the terms of the polynomial contained under or within it are to be considered as one quantity, and taken together. Thus,

$$(a+b-c)\,(3a-2b+c),$$

denotes that the trinomial within one parenthesis is to be multiplied by the trinomial within the other.

The vinculum is placed *under* the polynomial when the polynomial is the numerator of a fraction. Thus,

$$\frac{4b^2-3cd}{2b},$$

denotes that the difference between four times the square of b, and three times the product of c and d, is to be divided by twice b.

The vinculum is sometimes placed vertically. Thus,

$$\left.\begin{array}{r}+4a\\-5b\\+6c\end{array}\right|x$$

is the same as

$$(4a-5b+6c)x.$$

(25.) *Similar terms* are those which are composed of the same letters affected with the same exponents. Thus, in

$$9a^2b-3a^2+15a^2b+14a^2+5b^2c^2d+7b^2cd^2,$$

the terms $9a^2b$ and $+15a^2b$ are similar, and the terms $-3a^2$ and $+14a^2$ are similar; but

$$5b^2c^2d \text{ and } 7b^2cd^2$$

are not similar; for, although they are composed of the same letters, yet the same letters are not affected with the same exponents.

(26.) When a polynomial contains several similar terms, it may be reduced to a simpler form. Thus,

$$9a^2b-3a^2+15a^2b+14a^2=24a^2b+11a^2.$$

For, by (*Art.* 19), we may arrange the polynomial in this form,

$$9a^2b+15a^2b+14a^2-3a^2;$$

but $9a^2b+15a^2b$ reduce to $24a^2b$, and $14a^2-3a^2$ to $11a^2$. Hence

$$9a^2b+15a^2b+14a^2-3a^2=24a^2b+11a^2.$$

Therefore, for the reduction of similar terms of a polynomial, we have the following

RULE.

(27.) *Add together all the coefficients of the additive terms; form a single subtractive term in the same manner. Then subtract the less coefficient from the greater, and to the remainder prefix the sign of the greater coefficient, and annex the literal part.*

In the reduction of similar terms, the coefficients are the only quantities that are affected.

When the sum of the additive terms is equal to the sum of the subtractive terms, they cancel each other.

EXAMPLES.

1. Reduce the polynomial

$$3a^2b-ab^2-2ab^2+3ab^2$$

to its simplest form. *Ans.* $3a^2b$.

2. Reduce the polynomial

$$5a^4b+3a^2bc-7ab+17ab+2a^2bc-6a^4b$$

to its simplest form. *Ans.* $5a^2bc-a^4b+10ab$.

3. Reduce the polynomial

$$3x-2x+3y-7y+2x-3x+4y$$

to its simplest form. *Ans.* 0.

CHAPTER I.

ADDITION.

(28.) ADDITION in *Algebra* consists in connecting quantities together by means of their proper signs, and reducing the similar terms.

RULE.

Write the quantities to be added so that one similar term shall fall under another.

Reduce the similar terms, and to the result annex those terms which can not be reduced, giving to each its particular sign.

EXAMPLES.

1. Add together the polynomials

$9xy-4bc+7x^2$, $4xy-bc+3x^2$, $xy-7bc+4x^2+d^3$;

writing them so that one similar term shall fall under another, we have

$$\begin{array}{r} 9xy-\ 4bc+\ 7x^2 \\ 4xy-\ \ bc+\ 3x^2 \\ xy-\ 7bc+\ 4x^2+d^3 \\ \hline 14xy-12bc+14x^2+d^3 \end{array}$$ sum required.

2. Add together the polynomials

$17a^4b^2+9a^3b-3a^2$, $-14a^4b^2+7a^2-9a^3$, $-15a^3b+7a^4b^2$
$-a^3$;

writing them so that one similar term shall fall under another, we have

$$\begin{array}{rrrr}
17a^5b^2+ & 9a^3b & -3a^2 & \\
-14a^5b^2 & & +7a^2 & -\ 9a^3 \\
7a^5b^2 & -15a^3b & & -\ \ a^3 \\
\hline
10a^5b^2- & 6a^3b & +4a^2 & -10a^3
\end{array}$$

3. Add together the polynomials $2x+3a$, $4x+a$, $5x+8a$, $7x+2a$, and $x+a$. *Ans.* $19x+15a$.

4. Add together $7x^2-5bc$, $3x^2-bc$, x^2-4bc, $5x^2-bc$, and $4x^2-4bc$. *Ans.* $20x^2-15bc$.

5. Add together $7a^3-3a^2b+2ab^2-3b^3$, $ab^2-a^2b-b^3+4a^3$, $-5b^3+5ab^2-4a^2b+6a^3$, and $-a^2b+4ab^2-4b^3+a^3$. *Ans.* $18a^3-9a^2b+12ab^2-13b^3$.

6. Add together $2x^2y-x+2$, $x^2y-4x+3$, $4x^2y-3x+1$, and $5x^2y-7x+7$. *Ans.* $12x^2y-15x+13$.

7. Add together $2ab^4-18a^3b+6a^2b^2-8ab^4+7a^3b-5a^2b^2-5a^3b+6ab^4+11a^2b^2$. *Ans.* $12a^2b^2-16a^3b$.

8. Add together $3x-y-6z-115d-9m$, $6z-5m-d+6n-3y$, $3y-2x-3z+27n+11m$, $3n-7m+5x-8z+9d$, and $17z-6x-7y-2d-5n$.

Ans. $6z-8y-109d+31n-10m$.

9. Add together $5a^3-3x^2+3y$, $4a^3-x^2+4y$, a^3-7x^2+7y, $7a^3-x^2+y$, $8a^3-9x^2+9y$, and $7a^3-11x^2+y$.

Ans. $32a^3-32x^2+25y$.

10. Required the sum of $4x^2-3x+4$, $x-2x^2-5$, $1+3x^2-5x$, $2x-4+7x^2$, and $13-x^2-4x$.

Ans. $11x^2-9x+9$.

11. Add together $4x^3-2x+y$, $4x-y-x^3$, $9y+7x^3-x$, and $21x-2y+9x^3$. *Ans.* $19x^3+22x+7y$.

12. Add together $4x^3-3xy+3y-3-3x^2$, $5y^2+5x^3-3x^2+3xy+5y$, $30+6x^2+2x-3y^2-2x^3$, and $2x^2-8-5xy-7y-2y^2$.

Ans. $7x^3-5xy+y+19+2x^2+2x$.

13. Add together $6a^3b^2c-6adx^3+17a^4x$, $3a^3b^2c-16a^4x-9adx^3$, and $16adx^3-a^4x-8a^3b^2c$.

Ans. $a^3b^2c+adx^3$.

14. Add together $2a^2+3ab+8c^2+d^2$, $5a^2-7ab+5c^2-d^3$, and $4ab-2a^2+3c^2+30$.

Ans. $5a^2+16c^2+d^2-d^3+30$.

15. Add together $2a^2x-3abc^2+2b^3-3a^2$, $3b^3-2a^2x+a^2-5c^2$, $4c^3-2b^3+5abc^2+100$, and $20abc^2+16a^2x-bc-80$. *Ans.* [illegible]

16. Add together $8a^5-10a^4b-16a^3b^2+4a^2b^3-12a^4b+15a^3b^2+24a^2b^3-6ab^4-16a^3b^2+20a^2b^3+32ab^4-8b^5$.

Ans. $8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5$.

17. What is the sum of $8ax+5ab+3a^2b^2c^2$, $6a^2-18ax+10ab$, and $10ax-15ab-6a^2b^2c^2$?

Ans. $6a^2-3a^2b^2c^2$.

18. Add $3x-y+d$, $4a-x-3y$, $5xy+7ax+y^2$, $3a$ [illegible] $-2xy+4x^2$, and $5y+2d+5x$.

Ans. $7x+y+3d+3xy+10ax+4a+y^2+4x^2$.

(29.) When quantities with literal coefficients are to be added together, such as

$$ax,\ bx,\ cx,\ ax^2y,\ bx^2y,\ cx^2y,\ \&c.,$$

it may be done *by placing the coefficients of similar quantities one after another, with their proper signs under a vinculum, or in a parenthesis, and then connect them to the common quantity by the sign of multiplication.* Thus,

Ex. 1. What is the sum of $ax+by+cdz$, $bx+dy+ez$, and $cx+ey+fz$?

$$\begin{array}{c} ax+by+cdz \\ bx+dy+ez \\ cx+ey+fz \\ \hline (a+b+c)x+(b+d+e)y+(cd+e+f)z. \end{array}$$

Ex. 2. What is the sum of ax^3+bx^2+cx, and ex^3-dx^2-fx?

We shall have $\begin{cases} ax^3+bx^2+cx \\ ex^3-dx^2-fx \end{cases}$

$$\overline{(a+e)x^3+(b-d)x^2+(c-f)x.}$$

Ex. 3. What is the sum of px^2+qx, rx^2+sx, $-mx^2-nx$, and nx^2+mx?

Ans. $(p+r-m+n)x^2+(q+s-n+m)x.$

Ex. 4. Required the sum of $ax^4-bx^3+cx^2$, $bcx^2-acx^3+c^2x$, and ax^2+c-bx.

Ans. $ax^4-(b+ac)x^3+(c+bc+a)x^2+(c^2-b)x+c.$

Ex. 5. Required the sum of $4x^2+7(a+b)^2$, $4y^2-5(a+b)^2$, and $a^2-4x^2-3y^2-(a+b)^2$.

Ans. $a^2+y^2+(a+b)^2$ + 2

Ex. 6. What is the sum of $ax^2+bx+cy$, $a'x^2-b'x+c'y$, and $a''x^2+b''x-c''y$?

Ans. $(a+a'+a'')x^2+(b-b'+b'')x+(c+c'-c'')y.$

SUBTRACTION.

(30.) Subtraction in Algebra consists in finding the difference between two algebraic quantities.

Let it be required to subtract $3a$ from $8a$, the result will be $8a-3a=5a$;

and if $b-c$ is to be subtracted from a, the result will be $a-(b-c)$,

which is equal to $a-b+c$. For, since it is the difference between b and c that is to be taken from a, it is evident that if we subtract b from a, the remainder,

$$a-b,$$

is too little by the number of units contained in c; we must therefore add c to that remainder, in order to make it correct.

This will appear more evident from the following: thus, if it were required to subtract 9 from 12, the difference is $12-9=3$; and if $9-3$ were taken from 12, it is plain that the remainder would be greater by 3 than if the whole number 9 were subtracted; that is,

$$12-(9-3)=12-9+3=6.$$

(31.) Hence we have, for the subtraction of algebraic quantities, the following general

Rule.

Change the signs of all the terms in the subtrahend, or conceive them to be changed, and proceed as in addition.

Examples.

1. From $18xy$ subtract $13xy$. Here, changing the sign of $13xy$, it becomes $-13xy$, which being connected with $18xy$ with its proper sign, we have

$$18xy-13xy=(18-13)xy=5xy, \textit{ Ans.}$$

2. From $18xy$ subtract $-13xy$.

Changing the sign of $-13xy$, it becomes $+13xy$, which being connected to $18xy$ with its proper sign, we have

$$\begin{aligned}18xy+13xy&=(18+13)xy\\&=31xy, \textit{ Ans.}\end{aligned}$$

3. From $24a^2b^2+17c^2d^2$ subtract $15a^2b^2-5c^2d^2$, we have

$$\begin{array}{r}24a^2b^2+17c^2d^2\\ \underline{15a^2b^2-5c^2d^2}\\ 9a^2b^2+22c^2d^2.\end{array}$$

Note. It is best not to change the signs, but conceive them to be changed only.

4. From $5a+6b-c+3d$ subtract $2a-3b+2c-d+e-f$.

$$\begin{array}{l} 5a+6b-\ c+3d \\ \underline{2a-3b+2c-\ d+e-f} \\ 3a+9b-3c+4d-e+f, \text{ Ans.} \end{array}$$

$$\begin{array}{ll} \text{5. From} & 7a^2bc-3ab^2c+\ 8b^2c \\ \text{Take} & \underline{-5a^2bc-3ab^2c-\ 3b^2c} \\ & 12a^2bc \qquad\quad +11b^2c \end{array}$$

$$\begin{array}{ll} \text{6. From} & ax+by+cz+axyz \\ \text{Take} & \underline{a'x+b'y+c'z+a'xyz} \end{array}$$

$$(a-a')x+(b-b')y+(c-c')z+(a-a')xyz, \text{ Ans.}$$

(32.) As quantities in a parenthesis are considered as one quantity with respect to other symbols (*Art.* 24), the sign prefixed *affects them all. When this sign is negative, the signs of all those quantities must be changed in putting them into the parenthesis or in taking them out of it.*

7. From ax^3-bx^2+cx take bx^3-cx^2+dx, we have

$$ax^3-bx^2+cx-(bx^3-cx^2+dx)=ax^3-bx^2+cx-bx^3+cx^2-dx;$$

or, by changing the order of the terms (*Art.* 19),

$$ax^3-bx^3-bx^2+cx^2+cx-dx=(a-b)x^3-(b-c)x^2+(c-d)x.$$

When $-cx^2$ is subtracted from $-bx^2$, the result is $-bx^2+cx^2$, or $-(b-c)x^2$, because the sign *minus* prefixed to $(b-c)$ changes the signs of b and c.

8. From $bx^3+qx^2-rx+py^2$ take $ax^3-cx^2+mx-sy^2$.

If we write the quantities so that one similar term shall fall under another, we have

$$\begin{array}{r} bx^3+qx^2-\ rx+py^2 \\ ax^3-\ cx^2+mx-sy^2 \\ \hline (b-a)x^3+(q+c)x^2-(r+m)x+(p+s)y^2. \end{array}$$

When $+mx$ is subtracted from $-rx$, the result is $-rx-mx$; and as this means that the sum of rx and and mx is to be *subtracted*, that *negative* sum is to be expressed by

$$-(rx+mx) \text{ or } -(r+m)x.$$

For the same reason, if from a we subtract the polynomial $my^2+ny^2-aby^2-ry^2$, we shall have

$$a-(my^2+ny^2-aby^2-ry^2)=a-(m+n-ab-r)y^2;$$

or

$$a-(my^2+ny^2-aby^2-ry^2)=a-my^2-ny^2+aby^2+ry^2.$$

9. From $\quad 5b-3a+2c+5$
   Take $\quad -2b-8a+7c+7$
   $7b+5a-5c-2$ *Ans.*

10. From $\quad 6ab-3xy+4xz$
    Take $\quad 2ab+6xz+2xy$
    $4ab-5xy-2xy$ *Ans.*

11. From $\quad x^3+3x^2y+3xy^2+y^3$
    Take $\quad x^3-3x^2y+3xy^2-y^3$
    $6x^2y \quad +2y^3$ *Ans.*

12. From $\quad 3a-17b-10b+13a-3a$
    Take $\quad 6b-\ 8a-10b-\ 2a+3a$
    $11a-23b+15a-6a$ *Ans.*

13. From $x^2-2xy+y^2+(x^2-y^2)+2(xy-y^2)$
    Take $x^2+2xy-y^2+(x^2-y^2)-2(xy-y^2)$
    $-4xy-2y^2 \quad + \quad 4(xy-y^2)$ *Ans.*

14. From $2a^2+\ ax+\ x^2-12a^2x^2+20ax^2-\ 4x^3+6a^2x^3$
    Take $-a^2-3ax-2x^2+16a^2x^2+12ax^2-12x^3+2a^2x^3$
    Ans. $3a^2+4ax+3x^2-28a^2x^2+8ax^2+8x^3+4$

15. From $\quad ax^2+bxy+cy^2$
Take $\quad \underline{dx^2-hxy+hy^2}$

Ans.

16. From $\quad (x+y)\sqrt{x^2+y^2}+(a+b)(a+b)^2$
Take $\quad \underline{(x+y)\sqrt{x^2+y^2}-(a+b)(a+b)^2}$
2(a+b)

Ans.

17. From $\quad m^2n^2x^2-2px+q^2$
Take $\quad \underline{p^2q^2x^2-2rx+cq^2}$

Ans.

18. From $\quad a^2x^2y^2-m^2x^2+3cx-4x^2-9$
Take $\quad \underline{a^2x^2y^2-n^2x^2+c^2x+bx^2+3}$

Ans.

19. From $\quad ax^3+bx^2+cx+d$
Take $\quad \underline{a'x^3+b'x^2+c'x+d'}$

Ans.

20. From $\quad a(x+y)+c(x-y)-bxy$
Take $\quad \underline{4(x+y)-7(x-y)-cxy}$

Ans.

(33.) By using the parenthesis we may make polynomials undergo several transformations, which are of great utility in various algebraic calculations. Thus,

$$\begin{aligned} a^3-3a^2b+3ab^2-b^3 &= a^3-(3a^2b-3ab^2+b^3) \\ &= a^3-b^3-(3a^2b-3ab^2) \\ &= a^3+3ab^2-(3a^2b+b^3) \\ &= -(-a^3+3a^2b-3ab^2+b^3). \end{aligned}$$

(34.) *Addition* and *subtraction* in Algebra do not always mean increase and diminution respectively; for the algebraic sum of $+a$ and $-b$ is equal to $a-b$, and the algebraic difference is equal to $a+b$.

MULTIPLICATION.

(35.) Multiplication may be divided into three cases:

1. When the multiplicand and multiplier are both monomials.

2. When the multiplicand is a polynomial, and the multiplier a monomial.

3. When both multiplicand and multiplier are polynomials.

Case I.

(36.) *When the multiplicand and multiplier are both monomials.*

Let it be required to multiply $3a^2b$ by $5a^3b^2$.

The quantities may be written $3aab$ and $5aaabb$. Now, by arranging the factors, we shall have

$$3\times 5\times aaaaabbb=15a^5b^3.$$

If we consider the manner in which this result has been obtained, we shall find that any factor is repeated as many times in the product as it is a factor in both the multiplicand and multiplier, and that the *powers* of the same quantity are multiplied simply by adding their exponents. Hence, for the multiplication of a monomial by a monomial, we have the following

Rule.

Multiply the coefficient of the multiplicand by the coefficient of the multiplier for the coefficient of the product. Write after this last coefficient all the letters that enter into both factors, affecting each with an exponent equal to the sum of its exponents in both factors.

Remark. This rule is perfectly general whatever may be the exponents, whether they are ***whole***, ***fractional***, ***positive***, or ***negative*** numbers.

EXAMPLES.

1. $5abc \times 7abc = 35a^2b^2c^2$.
2. $17x^2y^2z \times 5x^3y^2z^2 = 85x^5y^4z^3$.
3. $20ab^2cd^3 \times 6bc^2x = 120ab^3c^3d^3x$.
4. $m^2x \times 15px = 15m^2px^2$.
5. $6a^{-2}b^3c^{\frac{1}{2}} \times 9a^3b^{-2}c^{\frac{3}{4}} = 54abc^{\frac{5}{4}}$.
6. Multiply $4a^2b^3cd$ by $3abc^2d^3$. *Ans.*
7. Multiply $12\sqrt{xy} \times 4b =$ *Ans.*
8. Multiply $\frac{11}{2}x^2y^2z^3 \times 6xy^3z =$ *Ans.*
9. Multiply $13a^2b^3c^2 \times 5abxy =$ *Ans.*
10. Multiply $4x^2z^2 \times \frac{1}{2}x^3yz =$ *Ans.*

(37.) If the exponents are letters, the same rule is to be observed. Thus,

11. $a^m \times a^n = a^{m+n}$.
12. $5a^{m-1} \times 6a = 30a^m$.
13. $7ax^{m+1} \times 4a^2x^m = 28a^3x^{2m+1}$.
14. $5x^my^n \times 4x^ny^m =$ *Ans.*
15. $20a^xb^y \times 5a^pb^qc^r =$ *Ans.*
16. Find the continued product of

$$5a^mb^pc^q \times 6a^nb^rc^s \times 2a^{m+n}b.$$

Ans.

CASE II.

(38.) *When the multiplicand is a polynomial, and the multiplier a monomial.*

Let it be required to multiply $a^2 - 2b$ by $3c$. The required product is equal to the difference between the

square of a and twice b, taken as many times as there are units in $3c$.

If we multiply a^2 by $3c$, the product $3a^2c$ is too great by the number of units in $2b$, taken $3c$ times. We must, therefore, subtract the product of $2b$ by $3c$ from $3a^2c$. Hence we derive the following

Rule.

Multiply each term of the multiplicand by the multiplier, beginning at the left hand; and these partial products, being connected by their respective signs, will be the product required.

Examples.

1. Multiply $x^2+2xy+y^2$
   by $5x^2$
   Product $5x^4+10x^3y+5x^2y^2$.
2. Multiply $a^3-3a^2b+3ab^2-b^3$
   by $6ab$
   Product $6a^4b-18a^3b^2+18a^2b^3-6ab^4$.
3. Multiply $a^m+4a^{m-1}b+6b^m$
   by $3ab$
   Product $3a^{m+1}b+12a^mb^2+18ab^{m+1}$.
4. Multiply $5xy+3x^2-2y^2$ by $12aby$.
   Ans.
5. Multiply $6ax-5by+7xy$ by $7abxy$.
   Ans.
6. Multiply $15a^2b+3ab^2-12b^3$ by $6ab$.
   Ans.
7. Multiply ax^3-bx^2+cx-d by x.
   Ans.

8. Multiply ax^m+by^n+c by $5ac$.
Ans.

9. Multiply $a+b-c-d$ by $abcd$.
Ans.

Case III.

(39.) *When both multiplicand and multiplier are polynomials.*

Let it be required to multiply $a-b$ by $c-d$. First multiplying a by c, the product is ac; but b should have been subtracted from a before the multiplication; b units have, therefore, been taken c times in the a, which ought not to have been so taken; hence b, taken c times, must be subtracted, and there results $ac-bc$, as the product of $a-b$ by c. But the multiplier was $c-d$, and not c; therefore the multiplicand has been taken d times too often; d times the multiplicand, which is of the same form as c times the multiplicand, must be subtracted from the product of $a-b$ by c, but the rule for subtraction (*Art.* 31) is to change the signs of the subtrahend. The result is, therefore, $ac-bc-ad+bd$. Hence, when both multiplicand and multiplier are polynomials, we have the following

$$\begin{array}{l} a-b \\ \underline{c-d} \\ ac-bc \\ \underline{\quad\;\; -ad+db} \\ ac-bc-ad+bd \end{array}$$

Rule.

Multiply every term of the multiplicand by each term of the multiplier in succession, affecting the product of any two terms with the sign plus when the signs are alike, and with the sign minus when

the signs are unlike; and the algebraic sum of these partial products will be the product required.

Examples.

1. Multiply $3a + 2b - 5c$
by $5a - 3b$

$$\begin{array}{lrrrrr} & 15a^2 & +10ab & -25ac & & \\ & & -9ab & & -6b^2 & +15bc \\ \hline \text{Product} & 15a^2 & +ab & -25ac & -6b^2 & +15bc. \end{array}$$

2. Multiply $a^3 - b^3$
by $a^3 + b^3$

$$\begin{array}{lrrr} & a^6 & -a^3b^3 & \\ & & +a^3b^3 & -b^6 \\ \hline \text{Product} & a^6 & & -b^6 \end{array}$$

(40.) *If the polynomial factors are homogeneous, their product will be homogeneous, and the degree of each term will be equal to the sum of the degrees of any two terms of the multiplicand and multiplier.*

3. Multiply $5a + 3b - 2c$
by $4d + 5e + 6f$

$$\begin{array}{l} 20ad + 12bd - 8cd \\ \quad + 25ae + 15be - 10ce \\ \quad\quad + 30af + 18bf - 12cf \\ \hline 20ad + 12bd + 25ae - 8cd + 15be + 30af - 10ce + 18bf - 12cf. \end{array}$$

If no two of the partial products are similar, there can be no reduction of the terms of the product. Therefore, *the whole number of terms in the product will be equal to the number of terms in the multiplicand multiplied by the number of terms in the multiplier.*

4. Multiply $x^2 + 3xy + y^2$ by $6x^2 - 3xy + y^2$.

Ans.

5. Multiply $5a^2x-6axy+7z^2$ by $6a^2x-7axy+8z^2$.

Ans.

6. Multiply $3a^2-2ab+5$ by $a^2+2ab-3$.

Ans.

7. Multiply $a^3-4a+2b$ by $3a-b$.

Ans.

8. Multiply $2x^2-3xy+4y^2$ by $5x^2-6xy-2y^2$.

Ans.

9. Multiply $3y^3+2x^2y^2+3x^3$ by $2y^3-3x^2y^2+5x^3$.

Ans.

10. Multiply b^2x^2-3ay by $6x-3y$.

Ans.

11. Multiply $7x^2-2y-9$ by $3x^2-11y$.

Ans.

12. Multiply $x^6+x^4+x^2$ by x^2-1.

Ans.

13. Multiply $x^4+x^2y^2+y^4$ by $x^4-x^2y^2+y^4$.

Ans.

14. Multiply $24x^2-2ax-35a^2$ by $2x-3a$.

Ans.

15. Multiply $3x-5yz+ab$ by $-5x^2+4yz-8ab$.

Ans.

16. Multiply $2x^3-3x^2y^2+5y^3$ by $3x^3+2x^2y^2+3y^3$.

Ans.

17. Multiply $2x^3+4x^2+8x+16$ by $3x-6$.

Ans.

18. Multiply $3x^2-5b^2+3c^2$ by x^2-b^2.

Ans.

19. Multiply a^m+b^m by a^n+b^n.

Ans. $a^{m+n}+a^nb^m+a^mb^n+b^{m+n}$.

20. Multiply $ax+by$ by $ax+cy$.

Ans. $a^2x^2+(ab+ac)xy+cby^2$.

21. Multiply $a^3+2a^2x+2ax^2+x^3$
by $\underline{a^3-2a^2x+2ax^2-x^3}$
$a^6 \qquad -x^6$, *Ans.*

22. Multiply $a+b+c$ by $a+b+c$.
Ans. $a^2+2ab+b^2+2ac+2bc+c^2$.

23. Multiply $a+b$ by $a+b$.

$$(a+b)\times(a+b)=(a+b)^2=a^2+2ab+b^2.$$

From which we deduce the following

THEOREM I.

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second

EXAMPLES.

1. $(x+y)(x+y)=x^2+2xy+y^2$.
2. $(3x+2a)(3x+2a)=9x^2+12ax+4a^2$.
3. $(9x+3y)^2=$ *Ans.*
4. $(8ax+7bc)^2=$ *Ans.*
5. $(5+\frac{1}{2})^2=$ *Ans.*
6. $(6+\frac{1}{4})^2=$ *Ans.*
7. Multiply $a-b$ by $a-b$.

$$(a-b)^2=(a-b)(a-b)=a^2-2ab+b^2.$$

From which we deduce the following

THEOREM II.

The square of the difference of two quantities is qual to the square of the first, minus twice the product of the first by the second, plus the square of the second.

Rule.

Divide the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient.

Annex to this coefficient all the letters that enter into the dividend and divisor, affecting each with an exponent equal to the excess of the exponent in the dividend over that in the divisor.

Remark. This rule is also general, whatever may be the exponents.

Divide a^4 by a, we have $a^{4-1}=a^3$.

" $a^4 \div a^2 = a^{4-2} = a^2$.

" $a^4 \div a^3 = a^{4-3} = a^1 = a$.

" $a^4 \div a^4 = a^{4-4} = a^0 = 1$.

" $a^4 \div a^5 = a^{4-5} = a^{-1} = \frac{1}{a}$.

" $a^4 \div a^6 = a^{4-6} = a^{-2} = \frac{1}{a^2}$.

" $a^m \div a^n = a^{m-n} = \frac{a^m}{a^n}$.

(43.) By inspecting the above we perceive *that* $a^0=1$, *and that a factor may be taken from the numerator into the denominator, or from the denominator into the numerator by changing the sign of its exponent.*

Examples.

1. Divide $24abc$ by $3b$. *Ans.* $8ac$.
2. Divide $39x^3y^2z$ by $13x^2z$. *Ans.* $3xy^2$.
3. Divide $1728a^3b^2c$ by $12abcd^2$. *Ans.* $144a^2bd^{-2}$.
4. Divide $14ab^2cd$ by $2a^2bc^2$. *Ans.* $7a^{-1}bc^{-1}d$.

5. Divide $57a^5bc^2$ by $3a^6b^2c^4$.

Ans. $19a^{-1}b^{-1}c^{-2}$, or $\dfrac{19}{abc^2}$.

6. Divide $42x^2y^2z^2$ by $2x^2y^2z^2$. *Ans.* 21.

7. Divide $-5a^2x^2y^2$ by $-7a^4x^2$. *Ans.* $\frac{5}{7}a^{-2}y^2$.

8. Divide $15axy^3$ by $-3ay$. *Ans.* $-5xy^2$.

CASE II.

(44.) *When the dividend is a polynomial, and the divisor a monomial.*

A polynomial divided by a monomial is effected by dividing each term of the polynomial by the monomial, and connecting the terms by their proper signs.

EXAMPLES.

1. Divide $7ax-42a^2x+56a^3y$ by $7a$.

$$\textit{Ans. } \frac{7ax-42a^2x+56a^3y}{7a}=x-6ax+8a^2y.$$

2. Divide $15a^{2m}-9a^{2m-1}b+6a^mb^m$ by $3a^m$.

$$\textit{Ans. } \frac{15a^{2m}-9a^{2m-1}b+6a^mb^m}{3a^m}=5a^m-3a^{m-1}b+2b^m.$$

3. Divide $15x^2y^2z^2-15xy^2+5xz^2$ by $5xy$.

$$\textit{Ans. } \frac{15x^2y^2z^2-15xy^2+5xz^2}{5xy}=3xyz^2-3y+\frac{z^2}{y}.$$

4. Divide $20a^3-15a^2+25a$ by $5a$.

Ans. $4a^2-3a+5$.

5. Divide $12x^2y-60xy^2-6xy$ by $6xy$.

Ans. $2x-10y-1$.

6. Divide $5mx^3+30m^2x^2+25m^3x^3$ by $5mx$.

Ans. $x^2+6mx+5m^2x^2$.

7. Divide $3xyzw+12ayz-9y^2z$ by $3yz$.
Ans. $xw+4a-3y$.

8. Divide $36a^2b^2+120a^2b+60ab$ by $12ab$.
Ans. $3ab+10a+5$.

9. Divide $8x^3y^3+12x^2y^2+16xy$ by $4xy$.
Ans. $2x^2y^2+3xy+4$.

10. Divide $72a^2b^2+120a^2b-12ab$ by $3a$.
Ans. $24ab^2+40ab-4b$.

11. Divide $ab+ac-a$ by a. *Ans.* $b+c-1$.

12. Divide $8x^{m+n}-10x^{n+1}y^n+12x^{n+1}y$ by $2x^n$.
Ans. $4x^m-5xy^n+6xy$.

13. Divide $6ax-18axz+24x$ by $6x$.
Ans. $a-3az+4$.

14. Divide $15mx-30amx+25m$ by $5m$.
Ans. $3x-6ax+5$.

CASE III.

(45.) *To divide a polynomial by a polynomial.*

The dividend may be regarded as the *product* of the divisor into the quotient, the quotient being as yet unknown. The highest power of any letter in this *product* is evidently formed by the multiplication of the highest power of the same letter in the divisor, by the highest power of that letter in the quotient.

Hence, *both the divisor and the dividend should be arranged according to the regular powers of some letter. Then divide the first term on the left of the dividend by the first term on the left of the divisor for the first term of the quotient.*

Multiply the whole divisor by this term of the quotient, and subtract the product from the dividend.

Divide the first term of the remainder by the first term of the divisor for a second term of the quotient;

multiply as before, and subtract the product from the first remainder.

Continue this process till the first term of the remainder can not be exactly divided by the first term of the quotient.

The remainder (if any) and divisor are then to be written in the form of a fraction, and connected to the quotient by the sign of the remainder.

Examples.

1. Divide $a^4-2a^2x^2+x^4$ by $a^2+2ax+x^2$.

$$
\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
a^4-2a^2x^2+x^4 & a^2+2ax+x^2 \\
\underline{a^4+2a^3x+a^2x^2} & \overline{a^2-2ax+x^2}, \textit{ Quotient.} \\
\quad -2a^3x-3a^2x^2+x^4 & \\
\quad \underline{-2a^3x-4a^2x^2-2ax^3} & \\
\qquad\qquad a^2x^2+2ax^3+x^4 & \\
\qquad\qquad a^2x^2+2ax^3+x^4. &
\end{array}
$$

2. Divide $a^{2m}-2a^mb^m+b^{2m}$ by a^m-b^m.

$$
\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
a^{2m}-2a^mb^m+b^{2m} & a^m-b^m \\
\underline{a^{2m}-\ a^mb^m} & \overline{a^m-b^m}, \textit{ Quotient.} \\
\quad -\ a^mb^m+b^{2m} & \\
\quad -\ a^mb^m+b^{2m} &
\end{array}
$$

3. Divide 1 by $1-x$.

$$
\begin{array}{l|l}
1\ldots & 1-x \\
\underline{1-x} & \overline{1+x+x^2+\dfrac{x^3}{1-x}}, \textit{ Ans.} \\
\quad x & \\
\quad \underline{x-x^2} & \\
\qquad x^2 & \\
\qquad \underline{x^2-x^3} & \\
\qquad\quad x^3 & \\
\qquad\quad \overline{1-x} &
\end{array}
$$

4. Divide $ax^5+ax^4+bx^4+ax^3+bx^3+cx^3+ax^2+bx^2+cx^2+bx+cx+c$ by ax^2+bx+c.

The terms of the dividend may be arranged as follows:

$$\begin{array}{r|l|l|l|l|l}
ax^5+a & x^4+a & x^3+a & x^2+b & x+c & ax^2+bx+c \quad \textit{Divisor.} \\
+b & +b & +b & +c & & \overline{x^3+x^2+x+1} \\
 & +c & +c & & &
\end{array}$$

$$\begin{array}{l}
ax^5+bx^4+cx^3 \\
\hline
\end{array}$$

$$\begin{array}{r|l|l}
ax^4+a & x^3+a & x^2 \\
+b & +b & \\
 & +c &
\end{array}$$

$$\begin{array}{l}
ax^4+bx^3+cx^2 \\
\hline
\end{array}$$

$$\begin{array}{r|l|l}
ax^3+a & x^2+b & x \\
+b & +c &
\end{array}$$

$$\begin{array}{l}
ax^3+bx^2+cx \\
\hline
ax^2+bx+c \\
ax^2+bx+c.
\end{array}$$

5. Divide $12x^4-192$ by $3x-6$.

Ans. $4x^3+8x^2+16x+32$.

6. Divide $6x^6-6y^6$ by $2x^2-2y^2$.

Ans. $3x^4+3x^2y^2+3y^4$.

7. Divide $1-5x+10x^2-10x^3+5x^4-x^5$ by $1-2x+x^2$.

Ans. $1-3x+3x^2-x^3$.

8. Divide $x^6-x^4+x^3-x^2+2x-1$ by x^2+x-1.

Ans. $x^4-x^3+x^2-x+1$.

9. Divide $x^3+5x^2y+5xy^2+y^3$ by $x^2+4xy+y^2$.

Ans. $x+y$.

10. Divide x^4+4y^4 by $x^2-2xy+2y^2$.

Ans. $x^2+2xy+2y^2$.

11. Divide $4x^3+4x^2-29x+21$ by $2x-3$.

Ans. $2x^2+5x-7$.

12. Divide $48x^3-76x^2y-64xy^2+105y^3$ by $2x-3y$.
Ans. $24x^2-2xy-35y^2$.

13. Divide x^5+y^5 by $x+y$.
Ans. $x^4-x^3y+x^2y^2-xy^3+y^4$.

14. Divide $36x^2y-63xy^2+20y^3$ by $12xy-5y^2$.
Ans. $3x-4y$.

15. Divide $52x^5-93x^4y-70x^3y^2+48x^2y^3-27xy^4$ by $13x^3-7x^2y+3xy^2$.
Ans. $4x^2-5xy-9y^2$.

16. Divide $14xy-21yz+7cy+6gx-9gz+3cg$ by $7y+3g$.
Ans. $2x-3z+c$.

17. Divide $\frac{1}{2}x^3+x^2+\frac{3}{8}x+\frac{3}{4}$ by $\frac{1}{2}x+1$.
Ans. $x^2+\frac{3}{4}$.

18. Divide $2x^{3n}-6x^{2n}b^n+6x^nb^{2n}-2b^{3n}$ by x^n-b^n.
Ans. $2x^{2n}-4b^nx^n+2b^{2n}$.

19. Divide $x^2+ax+bx+ab$ by $x+a$. *Ans.* $x+b$.

20. Divide $x^2-ax-bx+ab$ by $x-b$. *Ans.* $x-a$.

21 Divide $x^3-(a+2)x^2+(2a+b)x-2b$ by $x-2$.
Ans. x^2-ax+b.

22. Divide $10a^3+11a^2b-15a^2c-19abc+3ab^2+15bc^2-5b^2c$ by $5a^2+3ab-5bc$.
Ans. $2a+b-3c$.

23. Divide a^7-b^7 by $a-b$.
Ans. $a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6$.

24. Divide 1 by $1+x$.
Ans. $1-x+x^2-x^3+x^4-x^5+$, &c.

25. Divide 1 by $1-2x+x^2$.
Ans. $1+2x+3x^2+4x^3+5x^4+$, &c.

26. Divide x^3-2x^2-15x by $x-5$. *Ans.* x^2+3x.

27. Divide $a^2-b^2-c^2+2bc$ by $a-b+c$.
Ans. $a+b-c$.

28. Divide $-6x^4+96$ by $-3x+6$.
Ans. $2x^3+4x^2+8x+16$.

ON FACTORING.

(46.) When a polynomial is the product of two or more factors, it is often necessary to resolve it into the factors of which it is composed. This can frequently be done by inspection.

Select all the factors common to every term of the polynomial for one factor, and write what remains of each term within a parenthesis for the other factor.

EXAMPLES.

1. Take, for instance, the polynomial

$$ax^2+bx^2-cx^2.$$

We see immediately that x^2 is a factor common to all the terms; hence,

$$ax^2+bx^2-cx^2=(a+b-c)x^2.$$

2. Take, for a second example,

$$6b^3x-6bx^3.$$

We perceive that $6bx$ is common to both terms; hence,

$$6b^3x-6bx^3=6bx(b^2-x^2);$$

but b^2-x^2 is the difference of the squares of two quantities, which, by Theorem III., is equal to the product of the sum and difference of those quantities.

Therefore, $6b^3x-6bx^3=6bx(b+x)(b-x)$.

3. Resolve $x^2+ax+bx-cx$ into its component factors. *Ans.* $x(x+a+b-c)$.

4. Resolve $5a^2bc+10ab^2c+15abc^2$ into two factors. *Ans.* $5abc(a+2b+3c)$.

5. Decompose $3x^4y^6-6x^3y^5z+3x^2y^4z^2$ into three factors. *Ans.* $3x^2y^4(xy-z)(xy-z)$.

6. Decompose $3x^3y-3xy^3$ into three factors. *Ans.* $3xy(x+y)(x-y)$.

7. Decompose y^3-2y^2-15y into three factors.

Ans. $y(y+3)(y-5)$.

8. Decompose $3x^3+6x^2+3ax-15x$ into factors.

Ans. $3x(x^2+2x+a-5)$.

9. Decompose $3abc+12abx-9a^2b$ into factors.

Ans. $3ab(c+4x-3a)$.

10. Decompose $y^{n+1}-y^{n+2}+y^{n+3}-y^{n+4}$ into factors.

Ans. $y^n(y-y^2+y^3-y^4)$.

11. Decompose $a^2-b^2-c^2+2bc$ into two factors.

Ans. $(a+b-c)(a-b+c)$.

12. Decompose a^2x-x^3 into three factors.

Ans. $x(a+x)(a-x)$.

(47.) If we multiply $(x+a)$ by $(x+b)$, we shall have for the product

$$x^2+(a+b)x+ab.$$

Or, if we multiply $(x-a)$ by $(x-b)$, the product will be

$$x^2-(a+b)x+ab.$$

Hence, if we have a polynomial of the form

$$x^2+9x+20 \text{ or } x^2-9x+20,$$

in which the coefficient of the second term can be divided into two parts, such that their product shall be equal to the third term, it can always be factored.

Thus, $9=5+4$ and $5\times4=20$.

Therefore, $x^2+9x+20=(x+5)(x+4)$.

13. Decompose $x^2-9x+20$ into two factors.

Ans. $(x-5)(x-4)$.

14. Decompose $x^2+13x+42$ into two factors.

Ans. $(x+6)(x+7)$.

15. Decompose $x^2+8x+15$ into two factors.

Ans. $(x+3)(x+5)$.

16. Decompose $x^2-14x+45$ into two factors.

Ans. $(x-5)(x-9)$.

17. Decompose $x^2-10x+21$ into two factors.

Ans. $(x-3)(x-7)$.

If we multiply $(x-a)$ by $(x+b)$, the product is

$$x^2-(a-b)x-ab.$$

(48.) Where the sign of the middle term will be *minus*, or *plus*, according as a is greater than b, or less than b.

Hence, if we have a polynomial of the form

$$x^2+2x-15,$$

or $$x^2-2x-15,$$

in which the coefficient of the second term is the difference between two numbers, whose product is equal to the third term, it can always be factored.

Thus, $\quad 2=5-3$ and $5\times3=15$;

$$\therefore\ x^2+2x-15=(x+5)(x-3),$$

and $$x^2-2x-15=(x-5)(x+3).$$

18 Decompose x^2-x-30 into factors.

Ans. $(x+5)(x-6)$.

19 Decompose $x^2+6x-27$ into two factors.

Ans. $(x+9)(x-3)$.

20. Decompose x^2+6x-7 into two factors.

Ans. $(x+7)(x-1)$.

(49.) By the ordinary rule of division we might obtain the quotient of a^m-b^m divided by $a-b$, when some particular number is substituted for m; but we shall prove generally that a^m-b^m is always exactly divisible by $a-b$. Thus,

$$\begin{array}{l|l} a^m-b^m & a-b \\ \cline{2-2} & a^{m-1} \\ \underline{a^m-a^{m-1}b} & \\ \quad a^{m-1}b-b^m = \text{1st remainder.} & \end{array}$$

Dividing a^m by a, by the rule for exponents, we have a^{m-1} for the quotient. Multiplying the divisor by this quantity, and subtracting the product from the dividend, we have for a remainder $a^{m-1}b-b^m$, which may be put under the form

$$b(a^{m-1}-b^{m-1}).$$

Now, a^m-b^m will be exactly divisible by $a-b$, if $a^{m-1}-b^{m-1}$ be divisible by $a-b$; that is, if the difference of the same powers of two quantities is divisible by the difference of those quantities, then the difference of the powers of the next higher degree is also divisible by that difference.

But a^2-b^2 is exactly divisible by $a-b$, or

$$\frac{a^2-b^2}{a-b}=a+b;$$

hence a^3-b^3 is divisible by $(a-b)$; and since a^3-b^3 is divisible by $a-b$, a^4-b^4 must be divisible, and so on to any power m.

Dividing a^m-b^m by $a-b$, we have

$$\frac{a^m-b^m}{a-b}=a^{m-1}+a^{m-2}b+a^{m-3}b^2+a^{m-4}b^3+\ldots.+b^{m-1}.$$

Let $m=2$, 3, 4, 5, 6, &c., we shall have

$$\frac{a^2-b^2}{a-b}=a+b,$$

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2,$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3=(a^2+b^2)(a+b),$$

$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4,$$

&c., &c.

(50.) Hence, *the difference of the same powers of any two quantities is exactly divisible by the difference of those quantities.*

It may also be proved that *the difference of two even powers of the same degree is exactly divisible by the sum of their roots;* that is,

$$\frac{a^2-b^2}{a+b}=a-b,$$

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3,$$

$$\frac{a^6-b^6}{a+b}=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5,$$

&c., &c.

(51.) And that *the sum of two odd powers of the same degree is exactly divisible by the sum of their roots.*

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2,$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4,$$

&c., &c.

(52.) *If* m *be any whole number whatever,* 2m *will always be an even number, and* 2m+1 *an odd number.*

21. Decompose x^3-y^3 into two factors.

Ans. $(x-y)(x^2+xy+y^2)$.

22. Decompose x^4-y^4 into three factors.

Ans. $(x-y)(x+y)(x^2+y^2)$.

23 Decompose x^3-8y^3 into two factors.

Ans. $(x-2y)(x^2+2xy+4y^2)$.

24. Decompose $27x^3+1$ into two factors.

Ans. $(3x+1)(9x^2-3x+1)$.

25. Decompose $8x^3-1$ into two factors.

Ans. $(2x-1)(4x^2+2x+1)$.

26. Decompose $27x^3-27y^3$ into factors.

Ans. $27(x-y)(x^2+xy+y^2)$.

(53.) We should impress upon the minds of young students that the object of Algebra is not that of solving difficult questions only, but to instruct him in the *language of symbols*, a language which he will find to be the key to the higher branches of science, and ultimately to the profoundest mysteries of Nature.

Every algebraical result should, therefore, be interpreted, and transformed into an arithmetical quantity. To render the analyst expert, we shall here introduce a number of examples of transformation, which he should be compelled to verify by the substitution of different values for the several literal factors.

Examples.

1. Thus, $a^2-b^2=(a+b)(a-b)$.

If we put $a=\frac{1}{2}$, $b=\frac{1}{3}$, we shall have

$$a^2-b^2=\left(\frac{1}{2}\right)^2-\left(\frac{1}{3}\right)^2=\frac{1}{4}-\frac{1}{9}=\frac{5}{36},$$

and $(a+b)(a-b)=\left(\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{5}{6}\times\frac{1}{6}=\frac{5}{36}$.

2. Find the numerical value of $(x+a)(x+b)$, or its equal $x^2+(a+b)x+ab$, when $x=\frac{1}{3}$, $a=\frac{1}{4}$, $b=\frac{1}{5}$.

Ans. $\frac{14}{45}$.

3. Find the numerical value of $(ax-by)^2+(ay+bx)^2$,

or its equal $(a^2+b^2)(x^2+y^2)$, when $a=2$, $b=1$, $x=3$, $y=4$. *Ans.* 125.

4. Find the numerical value of the identical expressions

$$(p^2-aq^2)^2+a(2pq)^2=(p^2+aq^2)^2,$$

when $a=1$, $p=4$, $q=3$.

Ans. 625.

5. Find the numerical value of each of the identical expressions

$$(x+y+z)^2+(x-y)^2+(x-z)^2+(y-z)^2=3(x^2+y^2+z^2),$$

when $x=2$, $y=3$, $z=4$.

Ans. 87.

6. Find the numerical value of each of the identical expressions

$$x(x-1)(x-2)+9x(x-1)+18x+6=(x+1)(x+2)(x+3),$$

when $x=3$.

Ans. 120.

7. Find the numerical value of the expression

$$\sqrt{xy+2x\sqrt{xy-x^2}}+\sqrt{xy-2x\sqrt{xy-y^2}},$$

when $x=4$, $y=4$.

Ans. 8.

8. Find the numerical value of the expression

$$x\,\frac{(a-y)(a-z)}{(x-y)(x-z)}+y\cdot\frac{(a-x)(a-z)}{(y-x)(y-z)}+z\cdot\frac{(a-x)(a-y)}{(z-x)(z-y)},$$

when $x=2$, $y=3$, $z=4$, and $a=9$.

Ans. 9.

9. Find the numerical value of the expression

$$x(x-1)(x-2)+3x(x-1)+x,$$

when $x=-6$.

Ans. -216.

10. Find the numerical value of the expression

$$c \cdot \frac{a(a+b)-(a^2-b^2)}{a^2-b^2-a(a-b)+b^2},$$

when $a=2$, $b=4$, and $c=6$.

Ans. 18.

11. Find the numerical value of the expression

$$2a^2-3a\{5ab-\overline{3c-2a^2}(a^3-[5-a^2b^3]3ab-6a^2)-10\},$$

when $a=-2$, $b=3$, $c=4$.

Ans. 45032.

12. Find the numerical value of the polynomial

$$\overline{3a^2-2b}[5a-3c^2\{2a^2b-3c(2b^3-\overline{a^3+3bc})+18a\}-5b^3]$$
$$-2a^3,$$

when $a=2$, $b=-3$, $c=-4$.

Ans. 261794.

CHAPTER II.

OF ALGEBRAIC FRACTIONS.

(54.) ALGEBRAIC fractions differ in no respect from arithmetical fractions; that is, the denominator shows the number of parts into which a quantity is divided, and the numerator the number of parts taken.

There are three principles which form the bases of all operations in regard to fractions.

1. *To multiply a fraction by any number, either multiply the numerator, or divide the denominator of the fraction by that number.*

2. *To divide a fraction by any number, divide the numerator, or multiply the denominator of the fraction by that number.*

3. *If the numerator and denominator of a fraction be both multiplied, or both divided by the same number, the value of the fraction will not be changed.*

CASE I.

(55.) *To reduce a fraction to its simplest form.*

RULE.

Decompose the numerator and denominator into factors, then cancel the factors common to both.

EXAMPLES.

1. Reduce $\frac{x^2-4x-21}{x^2+8x+15}$ to its simplest form.

The numerator is equal to $(x-7)(x+3)$,
and the denominator $(x+5)(x+3)$;

$$\therefore \frac{x^2-4x-21}{x^2+8x+15}=\frac{(x-7)(x+3)}{(x+5)(x+3)}=\frac{x-7}{x+5}, \textit{ Ans.}$$

2. Reduce $\frac{x^2+5x+6}{x^2+7x+12}$ to its simplest form.

Ans. $\frac{x+2}{x+4}$.

3. Reduce $\frac{a^2-x^2}{a^3-x^3}$ to its simplest form.

Ans. $\frac{a+x}{a^2+ax+x^2}$.

4. Reduce $\frac{a^2x^2-x^4}{a^4-x^4}$ to its simplest form.

Ans. $\frac{x^2}{a^2+x^2}$.

5. Reduce $\frac{a^2+ax}{a^3+x^3}$ to its simplest form.

Ans. $\frac{a}{a^2-ax+x^2}$.

6. Reduce $\frac{x^2-y^2}{x^2-2xy+y^2}$ to its simplest form.

Ans. $\frac{x+y}{x-y}$.

7. Reduce $\frac{2a^2-16a-6}{3a^2-24a-9}$ to its simplest form.

Ans. $\frac{2}{3}$.

8. Reduce $\frac{3a^2b+3ab^2}{3a^2+6ab+3b^2}$ to its simplest form.

Ans. $\frac{ab}{a+b}$.

9. Reduce $\frac{14x^2-7xy}{10xz-5yz}$ to its simplest form.

Ans. $\frac{7x}{5z}$.

10. Reduce $\frac{6n^2-12n+6}{15n^2-30n+15}$ to its simplest form.

Ans. $\frac{2}{5}$.

11. Reduce $\frac{x^2-9x+20}{x^2-4x}$ to its simplest form.

Ans. $\frac{x-5}{x}$.

12. Reduce $\frac{x^2-5x}{x^2-6x}$ to its simplest form.

Ans. $\frac{x-5}{x-6}$.

13. Reduce $\frac{6x^2+7xy-3y^2}{6x^2+11xy+3y^2}$ to its simplest form.

Ans. $\frac{3x-y}{3x+y}$.

Here $6x^2+7xy-3y^2=(2x+3y)(3x-y)$,
and $6x^2+11xy+3y^2=(2x+3y)(3x+y)$.

Example 13 can best be solved by finding the greatest common measure. In fact, whenever the factors which compose the numerator and denominator are not readily perceived by the student, the best way to reduce a fraction to its simplest form will be to *find first the greatest common measure, and divide both terms of the fraction by it.*

(56.) The *greatest common measure* of two or more quantities is the same in Algebra as it is in Arith-

metic, viz., *the product of all the prime factors common to the quantities.*

TO FIND THE GREATEST COMMON MEASURE.

Having arranged the polynomial, with reference to one of its letters, divide the one which contains the highest power of the letter by the other, as in division; then divide the last divisor by the remainder arising from the preceding division. Continue this process till nothing remains, the last divisor will be the greatest common measure.

The demonstration will be found in a subsequent part of the work.

EXAMPLES.

1. Find the greatest common measure, and reduce the fraction $\frac{a^6-a^2x^4}{a^6+a^5x-a^4x^2-a^3x^3}$ to its simplest form.

We perceive that a^2 is a simple factor of the numerator, and a^3 of the denominator; therefore a^2 is the greatest common measure of these simple factors, and must be reserved for a factor of the greatest common measure of the other factors of the terms of the given fraction. We have then

$$a^2(a^4-x^4) \text{ and } a^3(a^3+a^2x-ax^2-x^3),$$

or
$$a^4-x^4 \text{ and } a^3+a^2x-ax^2-x^3,$$

as the polynomials whose greatest common measure we are to find.

$$\begin{array}{r|l}
a^4-x^4 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. & a^3+a^2x-ax^2-x^3 \\ \cline{2-2}
\underline{a^4+a^3x-a^2x^2-ax^3} & a-x \\
-a^3x+a^2x^2+ax^3-x^4 & \\
\underline{-a^3x-a^2x^2+ax^3+x^4} & \\
2a^2x^2-2x^4=\text{1st remainder.} &
\end{array}$$

Decomposing this remainder into two factors, we have $2x^2(a^2-x^2)$.

Rejecting $2x^2$, and dividing the last divisor by this remainder, we have

$$\begin{array}{l|l}
a^3+a^2x-ax^2-x^3 & a^2-x^2 \\ \cline{2-2}
\underline{a^3-ax^2} & a+x \\
\quad a^2x-x^3 & \\
\quad \underline{a^2x-x^3} & \\
\qquad 0 &
\end{array}$$

Multiplying the divisor, which gives no remainder, by a^2, we have $a^2(a^2-x^2)$ for the greatest common measure. Hence,

$$\frac{a^6-a^2x^4}{a^6+a^5x-a^4x^2-a^3x^3}=\frac{(a^6-a^2x^4)\div a^2(a^2-x^2)}{a^6+a^5x-a^4x^2-a^3x^3\div a^2(a^2-x^2)}$$

$$=\frac{a^2+x^2}{a^2+ax}, \textit{ Ans.}$$

2. Find the greatest common measure, and reduce the fraction $\frac{6x^2+7xy-3y^2}{6x^2+11xy+3y^2}$ to its lowest terms.

Ans.

3. Find the greatest common measure, and reduce the fraction $\frac{x^4-y^4}{x^6-y^6}$ to its lowest terms.

Ans. Com. meas. $=x^2-y^2$.

Red. frac. $=\frac{x^2+y^2}{x^4+x^2y^2+y^4}$.

4. Find the greatest common measure, and reduce the fraction $\frac{x^4-1}{x^5+x^3}$ to its lowest terms.

Ans. Com. meas. $=x^2+1$.

Red. frac. $=\frac{x^2-1}{x^3}$.

5. Find the greatest common measure, and reduce the fraction $\frac{x^3-b^2x}{x^2+2bx+x^2}$ to its simplest form.

Ans. Com. meas. $=x+b$.

Red. frac. $=\frac{x^2-bx}{x+b}$.

6. Find the greatest common measure, and reduce the fraction $\frac{x^3+x^2-12x}{x^3+4x^2+5x+20}$ to its simplest form.

Ans. Com. meas. $=x+4$.

Red. frac. $=\frac{x^2-3x}{x^2+5}$.

CASE II.

(57.) *To reduce a mixed quantity to the form of a fraction.*

A *mixed quantity* is one which is composed of an *entire* part and a fraction.

An *entire* quantity is one which has not the form of a fraction.

RULE.

Multiply the entire part by the denominator of the fraction, and to the product annex the numerator with the sign that connects them.

*If the sign is minus, and the numerator contains more than one term, it must be placed within a parenthesis, or all its signs must be changed.**

* The reason of the Rule is simply this: that when a fraction is preceded by the sign *minus*, it denotes that the whole fraction is to be subtracted. Now, when the numerator is a polynomial, the bar which separates it from the denominator is to be regarded as a vinculum or parenthesis; hence, when this vinculum is taken away, all the signs of the numerator must be changed (*Art.* 32).

EXAMPLES.

1. Reduce $3x-9-\dfrac{3x^2-32}{x+3}$ to the form of a fraction.

$$3x-9-\frac{3x^2-32}{x+3}=\frac{(3x-9)(x+3)-(3x^2-32)}{x+3}$$

$$=\frac{3x^2-27-(3x^2-32)}{x+3}=\frac{3x^2-27-3x^2+32}{x+3}=\frac{5}{x+3}, \textit{ Ans.}$$

It is always best to require the student to express the work, as above, before actually performing the operations.

2. Reduce $x-\dfrac{a^2+b^2}{x}$ to the form of a fraction.

Ans. $\dfrac{x^2-a^2-b^2}{x}$.

3. Reduce $3x^2-6+\dfrac{6-a^2}{7-y}$ to the form of a fraction.

Ans. $\dfrac{21x^2-36-3x^2y+6y-a^2}{7-y}$.

4. Reduce $10x-\dfrac{3x^2-8b^2}{x-b}$ to its simplest form.

Ans. $\dfrac{7x^2-10bx+8b^2}{x-b}$.

5. Reduce $y+\dfrac{a^2-y^2}{y}$ to the form of a fraction.

Ans. $\dfrac{a^2}{y^2}$.

6. Reduce $a+x+\dfrac{2x^2}{a-x}$ to the form of a fraction.

Ans. $\dfrac{a^2+x^2}{a-x}$.

7. Reduce $1+2x-\frac{x-3}{5x}$ to the form of a fraction.

Ans. $\frac{10x^2+4x+3}{5x}$.

8. Reduce $1+\frac{a^2-x^2}{a^2+x^2}$ to the form of a fraction.

Ans. $\frac{2a^2}{a^2+x^2}$.

9. Reduce $1-\frac{a^2-x^2}{a^2+x^2}$ to the form of a fraction.

Ans. $\frac{2x^2}{a^2+x^2}$.

10. Reduce $ab+cd+\frac{abc-c^2d-2cd^2}{c+2d}$ to the form of a fraction.

Ans. $\frac{2ab(c+d)}{c+2d}$.

11. Reduce $1-\frac{a^2-2ab+b^2}{a^2+b^2}$ to the form of a fraction.

Ans. $\frac{2ab}{a^2+b^2}$.

12. Reduce $x^2+2xy+y^2-\frac{x^3-3x^2y+3xy^2-y^3}{x+y}$ to the form of a fraction.

Ans. $\frac{2y(3x^2+y^2)}{x+y}$.

Case III.

(58.) *To reduce a fraction to an entire or mixed quantity.*

Rule.

Divide the numerator by the denominator for the entire part, place the remainder, if any, over the denominator for the fractional part, and unite the two by the sign of the remainder. If the sign of the remainder be minus, and it consists of several terms, the signs of the terms, except the first, must be changed when placed in the numerator.

Examples.

1. Reduce $\dfrac{x^2+2xy+y^2-p-q}{x+y}$ to a mixed quantity.

$$\begin{array}{l|l} x^2+2xy+y^2-p-q & x+y \\ \cline{2-2} \underline{x^2+\ xy\qquad\ } & x+y \\ \qquad\ xy+y^2 & \\ \qquad\ \underline{xy+y^2\quad\ } & \\ \qquad\qquad -p-q. & \end{array}$$

Here the remainder $-p-q$, when made the numerator of a fraction, comes within a parenthesis, and hence the sign must be changed.

That is, $-p-q=-(p+q)$.

Therefore $\dfrac{x^2+2xy+y^2-p-q}{x+y}=x+y-\dfrac{p+q}{x+y}$, *Ans.*

2. Reduce $\dfrac{a^2+2ab+b^2-a^2+b^4}{a+b}$ to a mixed quantity.

Ans. $a+b-\dfrac{a^2-b^4}{a+b}$.

3. Reduce $\dfrac{a^2+x^2}{a-x}$ to a mixed quantity.

Ans. $a+x+\dfrac{2x^2}{a-x}$.

4. Reduce $\frac{20a^2-10a+4}{5a}$ to a mixed quantity.

Ans. $4a-2+\frac{4}{5a}$.

5. Reduce $\frac{6x^2-ax}{3x+1}$ to a mixed quantity.

Ans. $2x-\frac{2x+ax}{3x+1}$.

6. Reduce $\frac{x^3-y^3}{x-y}$ to an entire quantity.

Ans. x^2+xy+y^2.

7. Reduce $\frac{9x^2-18x+8x^2y^2}{9x}$ to a mixed quantity.

Ans. $x-2+\frac{8xy^2}{9}$.

8. Reduce $\frac{3x^2-12ax+y-9x}{3x}$ to a mixed quantity.

Ans. $x-4a-3+\frac{y}{3x}$.

Case IV.

(59.) *To find the least common multiple of two or more quantities.*

Rule.

Decompose each quantity into factors, and, selecting the highest power of each, multiply them together, the result will be the least common multiple required.

Examples.

1. What is the least common multiple of 45864, 3780, 22050, and 54600?

By decomposing the quantities, we find

$$45864=2.2.2.3.3.7.7.13=2^3.3^2.7^2.13,$$
$$3780=2.2.3.3.3.5.7 \quad =2^2.3^3.5\;.7,$$
$$22050=2.3.3.5.5.7.7 \quad =2\;.3^2.5^2.7^2,$$
$$54600=2.2.2.3.5.5.7.13=2^3.3\;.5^2.7.13.$$

Hence the least common multiple is

$$2^3.3^3.5^2.7^2.13=3439800.$$

2. What is the least common multiple of

$a^3+3a^2b+3ab^2+b^3$, $a^2+2ab+b^2$, and a^2-b^2?

Here $a^3+3a^2b+3ab^2+b^3=(a+b)^3$

$a^2+2ab+b^2 \quad =(a+b)^2$

$a^2-b^2 \quad =(a+b).(a-b).$

Hence the least common multiple is

$$(a+b)^3\times(a-b)=a^4+2a^3b-2ab^3-b^4.$$

3. What is the least common multiple of $27a$, $15b$, $9ab$, and $3a^2$?

Here $27a=3.3.3.a=3^3.a,$

$15b=3.5.b \quad =3\;.5.b,$

$9ab=3.3.a.b=3^2.a.b,$

$3a^2=3.a.a \quad =3\;.a^2.$

Hence $3^3.5.a^2.b=135a^2b.$

Case V.

(60.) *To reduce fractions to their least common denominator.*

Rule.

Find the least common multiple of all the denominators of the given fractions (*Art.* 59) *for the common denominator.*

Divide this common denominator by the denominator of each fraction, separately, then multiply the

quotients by the respective numerators, and the products will be the numerators of the fractions required.

EXAMPLES.

1. Reduce $\frac{3a^2b}{4cx^2}$, $\frac{y}{2x}$, and $\frac{5x^2}{8ac^2}$ to fractions having a common denominator.

The least common multiple of $4cx^2$, $2x$, and $8ac^2$ is by (*Art.* 59) $8ac^2x^2$. Therefore,

$8ac^2x^2 \div 4cx^2 = 2ac \therefore 3a^2b \times 2ac = 6a^3bc$ = Num. of 1st,
$8ac^2x^2 \div 2x = 4ac^2x \therefore y \times 4ac^2x = 4ac^2xy =$ " 2d,
$8ac^2x^2 \div 8ac^2 = x^2 \therefore 5x^2 \times x^2 = 5x^4 =$ " 3d.

Hence $\frac{6a^3bc}{8ac^2x^2}$, $\frac{4ac^2xy}{8ac^2x^2}$, and $\frac{5x^4}{8ac^2x^2}$ are the fractions required.

2. Reduce $\frac{x}{2}$, $\frac{x-1}{3}$, and $\frac{x^2+2}{4}$ to fractions having a common denominator.

3. Reduce $\frac{a-3x}{4}$, $\frac{3a-5x}{5}$, and $\frac{3x+5y}{20}$ to a common denominator.

Ans. $\frac{5a-15x}{20}$, $\frac{12a-20x}{20}$, $\frac{3x+5y}{20}$.

4. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and $\frac{5x^2}{6ac}$ to a common denominator.

Ans. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, $\frac{5x^2}{6ac}$.

5. Reduce the fractions $\frac{b}{a+x}$, $\frac{a^2}{3}$, and $\frac{a^2+x^2}{a+x}$ to a common denominator,

Ans. $\frac{3b}{3(a+x)}$, $\frac{a^2(a+x)}{3(a+x)}$, $\frac{3(a^2+x^2)}{3(a+x)}$.

6. Reduce the fractions $\frac{x}{3}$, $\frac{x+1}{5}$, and $\frac{1-x}{1+x}$ to a common denominator.

Ans. $\frac{5x(1+x)}{15+15x}$, $\frac{3x^2+6x+3}{15+15x}$, $\frac{15-15x}{15+15x}$.

Note. Mixed quantities must be brought to a fractional form, and the whole numbers have unity for a denominator.

7. Reduce $\frac{3a}{4}$, $\frac{2x}{3}$, and $a+\frac{2x}{3a}$ to fractions having a common denominator.

Ans. $\frac{9a^2}{12a}$, $\frac{8ax}{12a}$, $\frac{12a^2+8x}{12a}$.

8. Reduce a, $\frac{7x}{9}$, $\frac{x^2+1}{x^2-1}$, and $a+\frac{2}{3}$ to fractions having a common denominator.

Ans. $\frac{9ax^2-9a}{9x^2-9}$, $\frac{7x^3-7x}{9x^2-9}$, $\frac{9x^2+9}{9x^2-9}$, $\frac{9ax^2-9a+6x^2-6}{9x^2-9}$.

Case VI.

(61.) *To add fractional quantities.*

Rule.

Reduce the fractions to the least common denominator; then add the numerators, and place their sum over the common denominator.

Examples.

1. What is the sum of $1+\frac{2b^2}{a^2-b^2}$, $\frac{a+b}{a-b}$, and $\frac{a-b}{a+b}$?

Reduce the mixed quantity to the form of a fraction;

thus, $$1+\frac{2b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}.$$

The least common denominator of the fractions is a^2-b^2, and the fractions themselves become

$$\frac{a^2+b^2}{a^2-b^2},\ \frac{a^2+2ab+b^2}{a^2-b^2},\text{ and }\frac{a^2-2ab+b^2}{a^2-b^2}.$$

Adding their numerators, we have

$$a^2+b^2+a^2+2ab+b^2+a^2-2ab+b^2;$$

reducing the similar terms, and placing the result over the common denominator, we find

$$1+\frac{2b^2}{a^2-b^2}+\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{3a^2+3b^2}{a^2-b^2},\ \textit{Ans.}$$

2. What is the sum of $\frac{2x}{3}$, $\frac{7x}{4}$, and $\frac{2x+1}{5}$?

Ans. $2x+\frac{49x+12}{60}$.

3. What is the sum of $\frac{a}{b}+\frac{c}{d}+\frac{e}{f}$?

Ans. $\frac{adf+bcf+bde}{bdf}$.

4. Add $\frac{1+x^2}{1-x^2}$ and $\frac{1-x^2}{1+x^2}$ together.

Ans. $\frac{2(1+x^4)}{1-x^4}$.

5. What is the sum of $\frac{1}{1+x}$ and $\frac{1}{1-x}$?

Ans. $\frac{2}{1-x^2}$.

6. What is the sum of $\frac{a}{bx}$, $\frac{c}{dx^2}$, and $\frac{e}{fx^2}$?

Ans. $\frac{adfx^2+bcfx+bde}{bdfx^2}$.

7. Add $x+\frac{27-9x}{4}+\frac{2x+5}{3}+\frac{29+4x}{12}$ together.

Ans. $\frac{130-3x}{12}$.

Case VII.

(62.) ***To subtract one fractional quantity from another.***

Rule.

Reduce the fractions to a common denominator; then subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference over the common denominator.

Examples.

1. From $3x+\frac{x}{b}$ subtract $x-\frac{x-a}{c}$.

Reducing the mixed quantities to the form of a fraction, we have

$$\frac{3bx+x}{b} \text{ and } \frac{cx-(x-a)}{c};$$

bringing them to a common denominator, and placing the sign of subtraction between them, we have

$$\frac{3bcx+cx}{bc}-\frac{bcx-(bx-ab)}{bc}.$$

Taking away the parenthesis from the numerator of the subtrahend, it becomes

$$\frac{3bcx+cx}{bc}-\frac{bcx-bx+ab}{bc}.$$

Subtracting the numerator of the subtrahend from that of the minuend, we find that

$$3x+\frac{x}{b}-\left(x-\frac{x-a}{c}\right)=\frac{2bcx+cx+bx-ab}{bc}$$

$$=2x+\frac{cx+bx-ab}{bc}, \textit{ Ans.}$$

2. From $3y+\frac{y}{a}$ subtract $y-\frac{y-a}{c}$.

Ans. $2y+\frac{cy+ay-a^2}{ac}$.

3. From $\frac{x+y}{2}$ subtract $\frac{x-y}{3}$.

Ans. $\frac{x+5y}{3}$.

4. From $\frac{an+bm}{bn}$ subtract $\frac{py+qx}{qy}$.

Ans. $\frac{anqy+bmqy-bpny-bnqx}{bnqy}$

5. From $\frac{a+b}{a-b}$ subtract $\frac{a-b}{a+b}$.

Ans. $\frac{4ab}{a^2-b^2}$.

6. From $\frac{a^2+b^2}{2}$ subtract $\frac{(a+b)^2}{4}$.

Ans. $\frac{(a-b)^2}{4}$.

7. From $3x+\frac{23x}{45}$ subtract $3x+\frac{2x}{5}$.

Ans. $\frac{x}{9}$.

8. From $\frac{1+x^2}{1-x^2}$ subtract $\frac{1-x^2}{1+x^2}$.

Ans. $\frac{4x^2}{1-x^4}$.

9. From $5x^2$ subtract $\frac{3x^2}{8}$.

Ans. $\frac{37x^2}{8}$.

10. From $\frac{x+y}{b}$ subtract $\frac{c}{d}$.

Ans. $\frac{dx+dy-bc}{bd}$.

11. From $\frac{x-y}{2b}$ subtract $\frac{2y-4x}{3c}$.

Ans. $\frac{3cx-3cy-4by+8bx}{6bc}$.

12. Find the difference between

$$\frac{a^2}{b^2} \text{ and } \frac{(x-y)(x+y)(x^2+y^2)}{x^4-y^4}.$$

Ans. Either $\frac{a^2-b^2}{b^2}$ or $\frac{b^2-a^2}{b^2}$.

Case VIII.

(63.) *To multiply fractional quantities together.*

Rule.

If any of the quantities to be multiplied are mixed, they must be reduced to a fractional form; then multiply the numerators together for a numerator and the denominators together for a denominator.

If there are any factors common to the numerator and denominator, they must be canceled before the multiplication.

Examples.

1. Multiply $1-\frac{3x-20}{x^2-6x}$ by $1-\frac{8x-42}{x^2-5x}$.

First, $1-\frac{3x-20}{x^2-6x}=\frac{x^2-6x-(3x-20)}{x^2-6x}=\frac{x^2-9x+20}{x^2-6x}$,

and $1-\frac{8x-42}{x^2-5x}=\frac{x^2-5x-(8x-42)}{x^2-5x}=\frac{x^2-13x+42}{x^2-5x}$.

Hence $\frac{x^2-9x+20}{x^2-6x}\times\frac{x^2-13x+42}{x^2-5x}$ expresses the product.

But the numerators and denominators may both be factored, and the expression becomes

$$\frac{(x-5)(x-4)}{x(x-6)}\times\frac{(x-6)(x-7)}{x(x-5)}.$$

Canceling the factors common to the numerator and denominator, we have

$$\frac{x-4}{x}\times\frac{x-7}{x}=\frac{x^2-11x+28}{x^2}, \textit{ Ans.}$$

2. What is the product of $\frac{x^2-a^2}{ab}$ and $\frac{x^2+a^2}{a+b}$?

Ans. $\frac{x^4-a^4}{a^2b+ab^2}$.

3. Required the product of $x+\frac{x+1}{a}$ and $\frac{x-1}{a+b}$.

Ans. $\frac{ax^2+x^2-ax-1}{a^2+ab}$.

4. Find the continued product of $\frac{a}{b}, \frac{m}{n}, \frac{p}{q}, \frac{x}{y}$.

Ans. $\frac{ampx}{bnqy}$.

5. Find the continued product of

$$\frac{a}{bx}, \frac{b}{cx^2}, \frac{c}{dx^3}, \frac{d}{ex^4}, \frac{e}{fx^5}.$$

Ans. $\frac{a}{fx^{15}}$.

6. Multiply $a+\dfrac{ax}{a-x}$ by $\dfrac{a^2-x^2}{x+x^2}$.

Ans. $\dfrac{a^2+a^2x}{x+x^2}$.

7. Multiply $\dfrac{x^4-b^4}{x^2+2bx+b^2}$ by $\dfrac{x^2+bx}{x-b}$.

Ans. x^3+b^2x.

8. Find the continued product of

$$\frac{a+b}{a-b},\ \frac{a-b}{a+b},\ \frac{a^2-b^2}{a+b},\ \frac{a^2+2ab+b^2}{a^2-b^2}.$$

Ans. $a+b$.

9. Multiply $\dfrac{a^2+b^2}{a^2-b^2}$ by $\dfrac{a-b}{a+b}$.

Ans. $\dfrac{a^2+b^2}{a^2+2ab+b^2}$.

10. Multiply $\dfrac{x^4-b^4}{x^2-2bx+b^2}$ by $\dfrac{x-b}{x^2+bx}$.

Ans. $\dfrac{x^2+b^2}{x}$.

11. Multiply $\dfrac{x^2-4}{x^2-1}$, $\dfrac{x^2-1}{7}$, and $\dfrac{x-2}{x+2}$ continually together.

Ans. $\dfrac{x^2-4x+4}{7}$.

12. Multiply $\dfrac{x^2+5x+6}{x^2+7x+12}$ by $\dfrac{x^2+9x+20}{x^2+11x+30}$.

Ans. $\dfrac{x+2}{x+6}$.

13. Multiply $\dfrac{x^2-8x+15}{x^2-12x+35}$ by $\dfrac{x^2-15x+56}{x^2-17x+72}$.

Ans. $\dfrac{x-3}{x-9}$.

14. Multiply $\dfrac{x^2+x-12}{x^2-13x+40}$ by $\dfrac{x^2+2x-35}{x^2+x-12}$.

Ans. $\dfrac{x+7}{x-8}$.

15. Multiply $\dfrac{x^2-a^2}{x^2+bx-ax-ab}$ by $\dfrac{x^2+bx+cx+bc}{x^2+cx+dx+cd}$.

Ans. $\dfrac{x+a}{x+d}$.

(64.) It is sometimes more convenient to multiply mixed quantities by mixed quantities, according to the rule of multiplication of polynomials, without reducing them to the form of fractions.

EXAMPLES.

1. Multiply $x^2-\frac{3}{4}x+1$ by $x^2-\frac{1}{2}x+1$.

$$\begin{array}{l}
x^2-\frac{3}{4}x+1 \\
\underline{x^2-\frac{1}{2}x+1} \\
x^4-\frac{3}{4}x^3+x^2 \\
\quad -\frac{1}{2}x^3+\frac{3}{8}x^2-\frac{1}{2}x \\
\qquad\qquad \underline{+\ x^2-\frac{3}{4}x+1} \\
x^4-\frac{5}{4}x^3+\frac{19}{8}x^2-\frac{5}{4}x+1=\text{Product.}
\end{array}$$

2. Multiply $x^2+\frac{1}{2}x$ by $x^2+\frac{3}{4}x$.

Ans. $x^4+\frac{5}{4}x^3+\frac{3}{8}x^2$.

3. Multiply $x^2-\frac{1}{2}x+\frac{2}{3}$ by $\frac{1}{3}x+2$.

Ans. $\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3}$.

4. Multiply $x^2-\frac{3}{4}x+1$ by $x^2-\frac{1}{2}x$.

Ans. $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$.

5. Multiply $x^3-\frac{1}{2}x^2+\frac{1}{3}x-1$ by $x^3+\frac{1}{2}x^2-\frac{1}{3}x+1$.

Ans. $x^6-\frac{1}{4}x^4+\frac{4}{3}x^3-\frac{10}{9}x^2+\frac{2}{3}x-1$.

6. Multiply $x^3+\frac{1}{2}x^2+\frac{1}{3}x+1$ by $x^3+\frac{1}{2}x^2+\frac{1}{3}x+1$.

Ans. $x^6+x^5+\frac{11}{12}x^4+\frac{7}{3}x^3+\frac{10}{9}x^2+\frac{2}{3}x+1$.

Case IX.

(65.) *To divide one fractional quantity by another.*

Rule.

If there are any mixed quantities, reduce them to a fractional form, then invert the divisor, and proceed as in multiplication.

Cancel all the factors that are common to the numerator and denominator previous to multiplying.

Examples.

1. Divide $1+\frac{n-1}{n+1}$ by $1-\frac{n-1}{n+1}$.

First, $$1+\frac{n-1}{n+1}=\frac{n+1+n-1}{n+1}=\frac{2n}{n+1},$$

and $$1-\frac{n-1}{n+1}=\frac{n+1-(n-1)}{n+1}=\frac{2}{n+1}.$$

Hence $$\frac{2n}{n+1}\div\frac{2}{n+1}=\frac{2n}{n+1}\times\frac{n+1}{2}.$$

Canceling the factors $n+1$ and 2, we have

$$\left\{1+\frac{n-1}{n+1}\right\}\div\left\{1-\frac{n-1}{n+1}\right\}=n.$$

2. Divide $\frac{2ax+x^2}{a^2-x^2}$ by $\frac{x}{a-x}$.

Ans. $\frac{2a+x}{a^2+ax+x^2}$.

3. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$.

Ans. $\frac{x-b}{6c^2x}$.

4. Divide $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a-b}-\frac{b}{a+b}$.

Ans. 1.

5. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

Ans. $\frac{x^2+b^2}{x}$.

6. Divide $\frac{x^2+5x-14}{x^2+7x+12}$ by $\frac{x^2-3x-10}{x^2+7x+12}$.

Ans. $\frac{x-7}{x+5}$.

7. Divide $\frac{a^6-x^6}{a^2-2ax+x^2}$ by $\frac{a^2+ax+x^2}{a-x}$.

Ans. a^3+x^3.

8. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$.

Ans. $\frac{9x-3}{x}$.

9. Divide $\frac{ab-bx}{a+p}$ by $\frac{ac-cx}{a+p}$.

Ans. $\frac{b}{c}$.

10. Divide m by $\frac{mx}{x^2-y^2}$.

Ans. $\frac{x^2-y^2}{x}$.

11. Divide $\frac{3x^2-3}{2a+2b}$ by $\frac{x^2-1}{2a^2+2ab}$. *Ans.* $3a$.

12. Divide $\frac{2x^2-7}{x+a}$ by $\frac{a^2}{x^2+2ax+a^2}$.

Ans. $\frac{(2x^2-7)(x+a)}{a^2}$.

13. Divide $\frac{a+1}{a-1}$ by $\frac{1+a}{1-a^2}$.

Ans. $-(1+a)$.

14. Divide $\frac{a+x}{a-x}+\frac{a-x}{a+x}$ by $\frac{a+x}{a-x}-\frac{a-x}{a+x}$.

Ans. $\frac{a^2+x^2}{2ax}$.

15. Divide $\frac{x^2-11x+28}{x^2}$ by $\frac{x^2-13x+42}{x^2-5x}$.

Ans. $\frac{x^2-9x+20}{x^2-6x}$.

16. Divide $\frac{x+x^2}{3a^2}$ by $\frac{2ax+2ax^2}{7}$.

Ans. $\frac{7}{6a^3}$.

17. Divide $\frac{15ab}{a-x}$ by $\frac{10ac}{a^2-x^2}$.

Ans. $\frac{3ab+3bx}{2c}$.

(66.) It is frequently more simple in practice to divide mixed quantities by mixed quantities, without reducing them to the form of a fraction, especially where the division will terminate.

18. Divide $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$ by $x^2-\frac{1}{2}x$.

$$\begin{array}{l|l}
x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x & x^2-\frac{1}{2}x \\
\cline{2-2}
x^4-\frac{1}{2}x^3 & x^2-\frac{3}{4}x+1 \\
\cline{1-1}
\quad -\frac{3}{4}x^3+\frac{11}{8}x^2 & \\
\quad -\frac{3}{4}x^3+\frac{3}{8}x^2 & \\
\cline{1-1}
\qquad\qquad x^2-\frac{1}{2}x & \\
\qquad\qquad x^2-\frac{1}{2}x. &
\end{array}$$

19. Divide $x^4+\frac{5}{4}x^3+\frac{3}{8}x$ by $x^2+\frac{3}{4}x$.

Ans. $x^2+\frac{1}{2}x$.

20. Divide $\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3}$ by $\frac{1}{3}x+2$.

Ans. $x^2-\frac{1}{2}x+\frac{2}{3}$.

21. Divide $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$ by $x^2-\frac{1}{2}x$.

Ans. $x^2-\frac{3}{4}x+1$.

22. Divide $x^6-\frac{1}{4}x^4+\frac{4}{3}x^3-\frac{10}{9}x^2+\frac{2}{3}x-1$
by $x^3+\frac{1}{2}x^2-\frac{1}{3}x+1$.

Ans. $x^3-\frac{1}{2}x^2+\frac{1}{3}x-1$.

23. Divide $x^6+x^5+\frac{11}{12}x^4+\frac{7}{3}x^3+\frac{10}{9}x^2+\frac{2}{3}x+1$
by $x^3+\frac{1}{2}x^2+\frac{1}{3}x+1$.

Ans. $x^3+\frac{1}{2}x^2+\frac{1}{3}x+1$.

24. Divide $x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2$ by $\frac{4}{3}x-2$.

Ans. $\frac{3}{4}x^3-\frac{1}{2}x^2+1$.

If we have a fraction of the form

$$\frac{x}{y}=a,$$

we may observe that $\frac{-x}{y}=-a$ or $\frac{x}{-y}=-a$, but $\frac{-x}{-y}=a$; that is,

(67.) *If the sign of the numerator or that of the denominator be changed, the sign of the quotient will be changed also; but if the signs of both terms be changed, the sign of the quotient will remain the same as before.*

Examples in Addition, Subtraction, Multiplication, and Division of Fractions.

1. Add $\frac{a^2-ab+b^2}{a-b}$ to $\frac{a^2+ab+b^2}{a+b}$.

Ans. $\frac{2a^3}{a^2-b^2}$.

2. Add $\dfrac{a-b}{m^2+mn+n^2}+\dfrac{a+b}{m^2-mn+n^2}$.

Ans. $\dfrac{2m(m^2+2n^2)}{m^4+m^2n^2+n^4}$.

3. Add $\dfrac{1}{4a^2(a+x)}$, $\dfrac{1}{4a^2(a-x)}$, and $\dfrac{1}{2a^2(a^2+x^2)}$ together.

Ans. $\dfrac{1}{a^4-x^4}$.

4. Find the algebraic sum of

$$\frac{7a-10}{5}+\frac{30-27a}{30}-\frac{3a-7}{6}.$$

Ans. $\dfrac{1}{6}$.

5. Multiply $\dfrac{3x^2}{5x-10}$ by $\dfrac{15x-30}{2x}$.

Ans. $\dfrac{9x}{2}$.

6. Multiply $\dfrac{2x-2n}{3bx}$ by $\dfrac{3nx}{5x-5n}$.

Ans. $\dfrac{2n}{5b}$.

7. Multiply $\dfrac{m^2-n^2}{m+b}\times\dfrac{m^2-b^2}{m+n}$ by $\dfrac{m}{mn-n^2}$.

Ans. $\dfrac{m(m-b)}{n}$.

8. Multiply $\dfrac{x^4-y^4}{x^2-n^2}\times\dfrac{x+n}{x^2+y^2}$ by $\dfrac{x-n}{x-y}$.

Ans. $x+y$.

9. Multiply $\dfrac{a^4+2a^3b+2ab^3+b^4}{a^2+ab+b^2}$ by $\dfrac{a^4-2a^3b+2ab^3-b^4}{a-b}$.

Ans. a^4+b^4.

10. Divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$.

Ans. $\frac{2a+x}{c^2+cx+x^2}$.

11. Divide $\frac{3a}{2a-2}$ by $\frac{2a}{a-1}$.

Ans. $\frac{3}{4}$.

12. Divide $\left(b+\frac{2b}{b-3}\right)$ by $\left(b-\frac{2b}{b-3}\right)$.

Ans. $\frac{b-1}{b+5}$.

13. Divide $\frac{3a^5-3ab^4}{4a^3b+8a^2b^2+4ab^3}$ by $\frac{3a^3b^2+3ab^4}{2ab^4+2b^5}$.

Ans. $\frac{(1-b)b}{2a}$.

14. Divide $\frac{5a^3-5ax^2}{6bc^2-6bx^2}$ by $\frac{a^2-ax}{bc+bx}$.

Ans. $\frac{5(a+x)}{6(c-x)}$.

15. Divide $\left(a+b+\frac{2b^2+2ab}{a-b}\right)$ by $\left(a-b-\frac{2ab-2b^2}{a+b}\right)$.

Ans. $\frac{(a+b)^2}{(a-b)^2}$.

16. Divide $\left(m+n+\frac{n^2}{m}\right)$ by $\left(m+n+\frac{m^2}{n}\right)$.

Ans. $\frac{n}{m}$.

17. Divide $n^4-\frac{1}{n^4}$ by $n-\frac{1}{n}$.

Ans. $n^3+n+\frac{1}{n}+\frac{1}{n^3}$.

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18. Divide $\dfrac{y^3+1}{xy^3}$ by $\dfrac{y^2-y+1}{y}$.

Ans. $\dfrac{y+1}{xy^2}$.

19. Find the algebraic sum of

$$\frac{5}{6}+\frac{3b-1}{24}-\frac{3b-5}{24}.$$

Ans. 1.

20. Add $\dfrac{(a+b)(a^2+b^2-c^2)}{ab}+\dfrac{(b+c)(b^2+c^2-a^2)}{bc}$

$+\dfrac{(a+c)(a^2+c^2-b^2)}{ac}$.

Ans. $2(a+b+c)$.

CHAPTER III.

EQUATIONS.

(68.) An *equation* is the algebraic enunciation of some particular problem.

It consists of two members separated by the sign of equality. The quantity on the left of the sign = is called the *first member;* the quantity on the right, the *second member.* A member may be composed of one term or more; the student should, therefore, early familiarize his mind to the distinction between a member and a term.

Equations are of different kinds.

1°. An equation may be such that one of its members is but a repetition of the other; as,

$$6ax-b=6ax-b,$$

or

$$(x-a)(x+a)=x^2-a^2.$$

Such equations are called *identical equations.*

An *identical equation* is one whose two members are either identical, or of such a form as to be reducible to identity by performing the operations indicated by them.

2°. The equation may be such that the equality subsisting between the two members can not be made evident, until, by certain transformations, the value of the unknown quantity becomes known. *When this value is ascertained the equation is solved.* And if this value of the unknown quantity, being substituted

for the unknown quantity itself in the original equation, makes the two members equal to each other, *the equation is said to be verified.*

Such equations as this latter kind are the objects of our investigations.

Equations are divided into *degrees*, according to the highest power of the unknown quantity which they contain.

Those which involve the first power only of the unknown quantity are called *simple equations*, or equations of the *first degree;* those into which the square of the unknown quantity enters are called *quadratic equations*, or equations of the *second degree;* and, universally, the degree of an equation is determined by the greatest exponent with which the unknown quantity is affected, without reference to other terms. Thus,

$ax+b-c=d+x$ is an equation of the first degree.

$ax^2+b-c=d+x$ is an equation of the second degree.

If more than one unknown quantity enter into an equation, its degree is determined by the greatest sum of the exponents with which the unknown quantities are affected in any of its terms when these exponents are not fractional. Thus,

$xy+bx=c$ is an equation of the second degree.

$x^2y+xy+b=c$ is an equation of the third degree.

Numerical equations are those which contain numbers only, with the exception of the unknown quantity. Thus, $x^2-3x=7$ is a numerical equation.

Literal equations are those in which a part or all of the known quantities are represented by letters.

$ax^2+bx-5=c+2$ and $ax+bx+cx=d+e$ are literal equations.

(69.) SOLUTION OF EQUATIONS OF THE FIRST DEGREE CONTAINING BUT ONE UNKNOWN QUANTITY.

In every equation the unknown quantity is united with the known quantities in one or more of the following different ways, viz., by *addition*, *subtraction*, *multiplication*, or *division*.

The various transformations which we make in an equation, in order to obtain the value of the unknown quantity, are founded upon the following well-known axioms:

1. *If the same quantity be added to equal quantities, the sums will be equal.*
2. *If the same quantity be subtracted from equal quantities, the remainders will be equal.*
3. *If equal quantities be multiplied by the same quantity, the products will be equal.*
4. *If equal quantities be divided by the same quantity, the quotients will be equal.*

Ex. 1. Let it be required to solve the equation

$$x+12=20,$$

which is the algebraic enunciation of the question, *What number is that which, being increased by* 12, *the sum shall be* 20?

If from the two equal quantities $x+12$ and 20 we subtract 12, the remainder, by *Ax.* 2, will be equal; but 12 taken from $x+12$ leaves x, and we shall have

$$x=20-12$$
$$=8, \text{ the value of } x \text{ required.}$$

Ex. 2. Let the equation be

$$x-12=20,$$

or the algebraic enunciation of the problem, *What*

number is that from which, if 12 *be subtracted, the remainder will be equal to* 20?

If to the two equal quantities $x-12$ and 20 we add 12, the sums, by *Ax.* 1, will be equal; hence we have

$$x=20+12$$
$$=32, \text{ the value of } x \text{ required.}$$

If we had taken the equations

$$x+a=b \quad . \quad . \quad . \quad (1)$$

and

$$x-a=b \quad . \quad . \quad . \quad (2)$$

the same principles would apply, and we would have from the 1st,

$$x=b-a,$$

and from the 2d, $x=b+a.$

(70.) Hence, *we may transpose any term of an equation from one member to the other by changing its sign.*

(71.) *If the same quantity appear in each member of the equation affected with the same sign, it may be canceled.*

Ex. 3. Let it be required to solve the equation

$$ax=b;$$

or, *What number is that which, being multiplied by* a, *the product will be* b?

If the two equal quantities ax and b be both divided by the same quantity a, the quotients, by *Ax.* 4, will be equal. Hence,

$$x=\frac{b}{a}, \text{ the value of } x \text{ required.}$$

If $b=36$ and $a=12$, we shall have $x=4$, the value required.

Ex. 4. Let it be required to solve the equation

$$\frac{x}{a}=b;$$

or, *What number is that which, being divided by* 12, *the quotient will be* 4, by putting $12=a$ and $4=b$?

Now, if the two equal quantities $\frac{x}{a}$ or $\frac{x}{12}$, and b or 4, be both multiplied by a or 12, the product, by *Ax*. 3, will be equal; hence

$$x=ab,$$

or $x=48$, the value required.

From which it follows that,

(72.) *When the unknown quantity is multiplied by a known quantity, its value is found by dividing both members of the equation by this known quantity.*

(73.) And *when the unknown quantity is divided by a known quantity, its value is found by multiplying both members by this quantity.*

Ex. 5. Let the equation to be solved be

$$\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}, \text{ to find the value of } x.$$

In order to solve this equation, we must clear it of fractions. To do this, reduce the fractions to their least common denominator (*Art.* 60), the equation becomes

$$\frac{3x-9}{6}+\frac{2x}{6}=\frac{120}{6}-\frac{3x-57}{6}.$$

Suppressing the denominator in each fraction, the result is

$$3x-9+2x=120-\overline{3x-57}.$$

The bar separating the numerator of the fraction

$\frac{3x-57}{6}$ from its denominator is a parenthesis, showing that the whole fraction is to be subtracted from the preceding quantity; and when this parenthesis is taken away by the removal of the denominator, we must change the signs of the terms in the numerator. We shall then have

$$3x-9+2x=120-3x+57.$$

Hence it appears that,

1°. *In order to clear an equation of fractions, find the least common multiple of all the denominators, then multiply every term of the equation by it.*

2°. *In multiplying a fractional term, divide the least common multiple by the denominator, and multiply the quotient by the numerator.*

3°. *When a fractional term is preceded by the sign minus, and its numerator is composed of more than one term, the signs of those terms must be changed when the denominator is removed.*

We shall now proceed with the solution.

Transposing the unknown terms into the first member, and the known terms into the second, we have

$$3x+2x+3x=120+57+9;$$

and, reducing the similar terms in both members, we find

$$8x=186;$$

$$\therefore\ x=23\tfrac{1}{4}, \text{ the value of } x \text{ required.}$$

(74.) Every equation having a numerical solution should be verified. *An equation is said to be verified when, by substituting the value of the unknown quantity, the first member becomes equal to the second.*

We find the value of x to be $23\frac{1}{4}$.

Substitute this value for x in the original equation, we have

$$\frac{23\frac{1}{4}-3}{2}+\frac{23\frac{1}{4}}{3}=20-\frac{23\frac{1}{4}-19}{2}$$

$$10\tfrac{1}{8}+7\tfrac{3}{4}=20-2\tfrac{1}{8}$$

$$17\tfrac{7}{8}=17\tfrac{7}{8},$$

the verification required.

Ex. 6. Let us take, as another example, the equation

$$3x-a+cx=\frac{a+x}{3}-\frac{b-x}{a}, \text{ to find the value of } x.$$

We find $3a$ to be the least common multiple of the denominator. Multiplying every term of the equation by this, we have

$$9ax-3a^2+3acx=a^2+ax-3b+3x.$$

Transposing all the unknown terms to the first member, and the known terms to the second, we have

$$9ax-ax+3acx-3x=a^2+3a^2-3b.$$

Reducing similar terms in both members,

$$8ax+3acx-3x=4a^2-3b.$$

Factoring the first member, we have

$$(8a+3ac-3)x=4a^2-3b.$$

Dividing both members of the equation by the multiplier or coefficient of x, we obtain

$$x=\frac{4a^2-3b}{8a+3ac-3}, \text{ the value required.}$$

(75.) From the preceding examples, to solve an equation of the first degree containing but one unknown quantity, we have the following

RULE.

If necessary, clear the equation of fractions, then

transpose all the unknown terms into the first member, and the known terms into the second member. Reduce both members to their simplest form, and divide by the coefficient or multiplier of the unknown quantity.

EXAMPLES.

1. Given $19x+13=59-4x$ to find the value of x.

Ans. $x=2$.

2. Given $4x+7-x=x+18$ to find the value of x.

Ans. $x=5\frac{1}{2}$.

3. Given $2x-8\frac{3}{4}=\frac{65}{4}-\frac{x}{2}$ to find the value of x.

Ans. $x=9$.

4. Given $5-\frac{x+4}{11}=x-3$ to find the value of x.

Ans. $x=7$.

5. Given $3x-\frac{11x-37}{2}=5-\frac{2x+6}{5}$ to find the value of x.

Ans. $x=7$.

6. Given $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$ to find the value of x.

Ans. $x=5$.

7. Given $7x+13\frac{3}{4}-\frac{x}{2}=\frac{4x}{5}-8\frac{3}{4}+\frac{41x}{5}$ to find the value of x.

Ans. $x=9$.

8. Given $\frac{x+1}{3}-\frac{x+4}{5}=16-\frac{x+3}{4}$ to find the value of x. *Ans.* $x=41$.

9. Given $\frac{x-1}{7}=7-\frac{23-x}{5}-\frac{4+x}{4}$ to find the value of x.

Ans. $x=8$.

10. Given $\frac{3x+4}{5}-\frac{7x-3}{2}=\frac{x-16}{4}$ to find the value of x.

Ans. $x=2$.

11. Given $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$ to find the value of x.

Ans. $x=4$.

12. Given $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$ to find the value of x.

Ans. $x=6$.

13. Given $\frac{4x-21}{9}+\frac{57-3x}{4}=\frac{949}{4}-\frac{5x-96}{12}-11x$ to find the value of x.

Ans. $x=21$.

14. Given $21-\frac{97-7x}{2}=\frac{5x-5}{8}-\frac{3x-11}{16}$ to find the value of x.

Ans. $x=9$.

15. Given $23+\frac{5x-1}{11}+\frac{3x-2}{5}-\frac{11x-3}{12}=\frac{13x-15}{3}-\frac{8x-2}{7}$ to find x.

Ans. $x=9$.

16. Given $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=77$ to find the value of x.

Ans. $x=60$.

17. Given $20-\frac{x}{2}=\frac{20-x}{6}-2x$ to find the value of x.

Ans. $x=-10$.

18. Given $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}+\frac{x}{d}=f$ to find the value of x.

Ans. $x=\frac{abcdf}{bcd+acd+abd+abc}$.

19. Given $\frac{x}{a}-\frac{dx}{c}+3ab=1$ to find the value of x.

Ans. $x=\frac{ac-3a^2bc}{c-ad}$.

20. Given $\frac{ax}{b}+\frac{cx}{d}+e=fx+\frac{gx}{h}+m$ to find x.

Ans. $x=\frac{bdhm-bdeh}{adh+bch-bdfh-bdg}$.

21. Given $\frac{a}{x}=\frac{b}{c}+\frac{d}{e}$ to find the value of x.

Ans. $x=\frac{ace}{be+cd}$.

22. Given $2ax-bx+2ab=4a^2-ab-3ax$ to find the value of x.

Ans. $x=\frac{4a^2-3ab}{5a-b}$.

23. Given $\frac{ax}{a-b}+4b=\frac{cx}{3a+b}$ to find the value of x.

Ans. $x=\frac{8ab^2+4b^2-12a^2b}{3a^2+ab-ac+bc}$.

24. Given $\frac{13a-7x}{a+b}+\frac{4a-x}{a-b}=\frac{a+b}{a-b}-cx$ to find x.

Ans. $x=\frac{11ab-16a^2+b^2}{6b-8a+a^2c-b^2c}$.

25. Given $\frac{x+ax-bx}{a-b}=\frac{cx-d}{c}$ to find x.

Ans. $x=\frac{d(b-a)}{c}$.

26. Given $ax-\frac{a^2-3bx}{a}-ab^2=bx+\frac{6bx-5a^2}{2a}-\frac{bx+4a}{4}$ to find x.

Ans. $x=\frac{2a(2b^2-5)}{4a-3b}$.

27. Given $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}+\frac{g}{hx}=k$ to find the value of x.

Ans. $x=\frac{adfh+bcfh+bdeh+bdfg}{bdfh}$.

28. Given $\frac{ax}{m}+\frac{b}{m}+\frac{cx}{k}=\frac{px-q}{m}-\frac{rx}{k}$ to find the value of x.

Ans. $x=\frac{bk+kq}{pk-ak-mr-cm}$

29. Given $\frac{(a+b)x}{a-b}+\frac{x}{a^2-b^2}=\frac{x+1}{a+b}$ to find the value of x.

Ans. $x=\frac{a-b}{(a+b)^2-(a-b-1)}$.

30. Given $\frac{x+1}{2}=16-\frac{x+2}{3}-\frac{x+3}{4}$ to find the value of x.

Ans. $x=13$.

31. Given $bx-ac=\frac{a^2x}{b-c}$ to find the value of x.

Ans. $x=\frac{abc-ac^2+bcd-c^2d}{b^2-bc-a^2}$.

32. Given $6-\frac{16+4x}{5}=\frac{3x+9}{2}-\frac{7x+5}{3}$ to find the value of x. *Ans.* $x=1$.

33. Given $2x-\frac{9x+20}{36}=16-\frac{4x-12}{36}-\frac{x}{4}$ to find x.

Ans. $x=8$.

In practice, however, it will sometimes be found more convenient to multiply first by a multiple of one or two of the denominators, and the equation may then be susceptible of great reduction before any further multiplication is necessary.

Take the equation

$$\frac{2x+\frac{17}{2}}{9}-\frac{13x-2}{17x-32}=\frac{x}{4}-\frac{x+16}{36}.$$

Multiply first by 36, then

$$8x+34-\frac{36(13x-2)}{17x-32}=9x-x-16;$$

$$\therefore\ 50=\frac{36(13x-2)}{17x-32},$$

or

$$25=\frac{18(13x-2)}{17x-32},$$

so that we have but the binomials $17x-32$ and $13x-2$ to multiply by 25 and 18, respectively,

$$25(17x-32)=18(13x-2),$$

or

$$425x-800=234x-36;$$

$$\therefore\ x=4.$$

(76.) *Any proportion may be converted into an equation by putting the product of the means equal to the product of the extremes.*

34. Given $\frac{10+x}{5}:\frac{4x-9}{7}::14:5$.

Multiplying the means together, we have

$$8x-18.$$

Multiplying the extremes together, we have

$$10+x.$$

These two products are equal to each other; therefore,

$$8x-18=10+x$$
$$7x=28$$
$$x=4, \text{ the value required.}$$

Verification:

$$\frac{14}{5}:1::14:5$$
$$14=14.$$

35. Given $\frac{17-4x}{4}:\frac{15+2x}{3}-2x::5:4$ to find the value of x.

Ans. $x=3$.

36. Given $\frac{5x+4}{2}:\frac{18-x}{4}::7:4$ to find the value of x.

Ans. $x=2$.

37. Given $x+4:x-11::5:2$ to find the value of x.

Ans. $x=21$.

38. Given $\frac{x-5}{4}:(x-5)::8:9$ to find the value of x.

Ans. $x=5$.

39. Given $x+6:38-x::4\frac{1}{2}:1$ to find the value of x.

Ans. $x=30$.

40. Given $3x-1:2x+1::3:1$ to find the value of x.

Ans. $x=-\frac{4}{3}$.

41. Given $3a : x :: b+5 : x-9$ to find the value of x.

$$Ans.\ x=\frac{27a}{3a-b-5}.$$

(77.) THE SOLUTION OF PROBLEMS PRODUCING EQUATIONS OF THE FIRST DEGREE, INVOLVING BUT ONE UNKNOWN QUANTITY.

Having thus far explained the method of solving an equation of the first degree containing one unknown quantity only, we shall proceed to point out the manner in which a given problem may be put into an equation. We have already observed that an equation is the algebraic enunciation of a problem, and, therefore, all that is necessary for the student is to translate the problem into *algebraic language.*

Every problem includes in its enunciation a certain number of conditions, either *explicit* or *implicit*, of such a kind that, taking the value of the unknown quantity, these conditions will be fulfilled. Now the value of this unknown quantity can be represented by a letter, and the arithmetical operations of the conditions of the question can be as readily *indicated by signs as performed numerically*, and the indication here spoken of constitutes the required equation. To become expert, however, in this translation requires reflection and practice. We shall give as a guide to the student the following

RULE.

Represent the unknown quantity by any one of the final letters of the alphabet; then, by means of the algebraic signs, perform the same operations as would

be necessary to verify its value if that value was already determined.

PROBLEMS.

1. Four merchants entered into partnership, the amount of capital being \$4755. B paid three times as much as A; C paid as much as A and B; and D paid as much as B and C. What did each pay?

Here, if we knew how much A paid, the sum paid by each of the others could easily be ascertained.

Let us, therefore, put x=the No. of dolls. paid by A,

then $3x=$ " " " B,

$4x=$ " " " C,

and $7x=$ " " " D.

But, by the problem, the whole amount paid was \$4755.

Hence $$x+3x+4x+7x=4755,$$

or $$15x=4755;$$

$\therefore\ x=$ 317=No. of dollars A paid,

951= " " B "

1268= " " C "

2219= " " D "

results which are readily verified.

2. Two pedestrians start from the same point and travel in the same direction; the first steps twice as far as the second, but the second makes 5 steps while the first makes but one. At the end of a certain time they are 300 feet apart. How far has each traveled?

Put x=the distance the first traveled; then, as the second takes 5 steps to his one, he would travel five times as far, if his steps were of equal length with

those of the first, but they are only half as long; hence the distance traveled by the second is

$$\frac{5x}{2}.$$

Now the difference between these two distances is equal to the distance they are apart; therefore,

$$\frac{5x}{2}-x=300.$$

Clearing fractions, $5x-2x=600$,

reducing $3x=600$,

$x=200=$distance 1st goes,

and $\frac{5x}{2}=500=$ " 2d goes,

results which are readily verified.

3. Given the sum of two numbers equal to s, and their difference equal to d, to find the numbers.

Let $x=$the less,

then $x+d=$the greater,

and $x+x+d=s$, by the question;

or $2x=s-d$

$x=\frac{1}{2}s-\frac{1}{2}d=$the less,

and $\frac{1}{2}s-\frac{1}{2}d+d=\frac{1}{2}s+\frac{1}{2}d=$greater.

As the results above are independent of any particular values attributed to s and d, it follows that,

(78.) *The greater of two quantities is equal to half the sum increased by half the difference; and*

The less is equal to half the sum diminished by half the difference.

These are called *formulas;* that is, *algebraic enunciations of particular rules.*

If we put $s=20$ and $d=8$, the two formulas become

$\frac{1}{2}s+\frac{1}{2}d=10+4=14=$greater,

$\frac{1}{2}s-\frac{1}{2}d=10-4=\ 6=$less.

4. A fox, pursued by a greyhound, is 60 of his own leaps in advance of the dog. He makes 9 leaps while the greyhound makes but 6; but 3 leaps of the greyhound are equal to 7 leaps of the fox. How many leaps must the greyhound make before he overtakes the fox?

It is manifest, from the enunciation of the problem, that the distance which must be traversed by the greyhound is composed of the 60 leaps which the fox is in advance, together with the space which the fox passes over from the time the greyhound starts until he overtakes him.

Let $x=$the number of leaps made by the greyhound.

Then, since the fox makes 9 leaps while the greyhound makes 6, it follows that the fox will make $\frac{9}{6}$ or $\frac{3}{2}$ leaps during the time the dog makes 1, and therefore will make $\frac{3x}{2}$ leaps while the greyhound makes x leaps. We might now suppose that, in order to obtain the equation required, it would be sufficient to put x equal to $60+\frac{3x}{2}$; but in so doing we would commit a manifest error, for the leaps of the greyhound are greater than those of the fox, and we should thus be equating two members related to a different unit. In order to remove this difficulty, we must either express the leaps of the fox in terms of those of the greyhound, or the contrary. By the equation, 3 leaps of the greyhound are equal to 7 leaps of the fox; hence 1 leap of the

greyhound is equal to $\frac{7}{3}$ leaps of the fox, and, consequently, x leaps of the greyhound are equal to $\frac{7x}{3}$ leaps of the fox. We have, therefore, the equation

$$\frac{7x}{3}=60+\frac{3x}{2}.$$

Clearing of fractions, it becomes

$$14x=360+9x,$$

and $$x=72.$$

Hence the greyhound will make 72 leaps before he reaches the fox, and in that time the fox will make $72\times\frac{3}{2}=108$ leaps.

Verification:

The 72 leaps of the greyhound are equal to $\frac{72\times7}{3}$ $=168$ leaps of the fox $=$ the whole distance. And $60+108=168=$ the number of leaps which the fox made from the beginning.

5. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days. How many days would the man be in drinking it alone?

Let $x=$the number of days required.

Now, as the man can drink it in x days, he can drink the $\frac{1}{x}$th part of it in a day.

The man and woman can drink the $\frac{1}{12}$th part of it in one day, and the woman $\frac{1}{30}$th part in one day.

If we subtract the part that the woman can drink ı a day from the part the man and woman both can rink in one day, the remainder will evidently be the art the man can drink in one day. We shall, there-ore, have two expressions for the same quantity, which nay be put equal to each other. Thus,

$$\frac{1}{12}-\frac{1}{30}=\frac{1}{x}.$$

Clearing the equation of fractions, we have

$$30x-12x=360,$$

or

$$18x=360$$

$$x=20 \text{ number of days required.}$$

Verification:

The man will drink the $\frac{1}{20}$th part in one day.

Man and woman, $\frac{1}{12}$th.

Woman, $\frac{1}{30}$th;

$\therefore \frac{1}{12}-\frac{1}{30}=\frac{5}{60}-\frac{2}{60}=\frac{3}{60}=\frac{1}{20}=$the part the man can drink in one; therefore it will take him 20 days.

6. What number is that whose fourth part exceeds its fifth part by 32?

Ans.

7. What number is that from which, if 8 be subtracted, three fourths of the remainder will be 60?

Ans. 88.

8. Two persons, A and B, lay out equal sums of money in trade; A gains $630, and B loses $435; A's money is now double of B's. What did each lay out?

Ans. $1500.

9. Two persons, A and B, have both the same income; A saves $\frac{1}{5}$th of his yearly; but B, by spending \$250 per annum more than A, at the end of 4 years finds himself \$500 in debt. What is their income?

Ans. \$625.

10. A footman agreed to serve his master for £8 a year and a livery, but was turned away at the end of 7 months, and received only £2 13*s.* 4*d.* and his livery. What was its value?

Ans. £4 16*s.*

11. A person in play lost $\frac{1}{4}$th of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$d of what he then had, and this done, found that he had but 12 shillings remaining. What had he at first?

Ans. 20

12. A hogshead containing 120 gallons was filled with a mixture of brandy, wine, and water. There were 10 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Ans. Brandy 25 gallons, wine 35 gallons, and water 60 gallons.

13. One carpenter, 12 journeymen, and four apprentices receive at the end of a certain time \$72. The carpenter received \$1 per day, each journeyman half a dollar, and each apprentice 25 cents. How many days were they employed?

Ans. 9 days.

14. In a certain orchard, $\frac{1}{3}$d are apple-trees, $\frac{1}{4}$th peach-trees, $\frac{1}{6}$th plum-trees, $\frac{1}{12}$th cherries, and 200 pear-trees. How many trees are in the orchard?

Ans. 1200.

15. A merchant finds that he has gained by speculation 15 per cent. on his capital, and that by this means it has become $15,571. What was his capital?

Ans. $13,540.

16. A farm of 864 acres is to be divided among three persons, A, B, and C, so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together. How many acres does each receive?

Ans. A 135, B 297, C 432 acres.

17. After paying away the fourth and fifth part of my money, I had $2.75 left. How much had I at first?

Ans. $5.

18. A person spent the $\frac{1}{3}$d part of his income for board and lodging, $\frac{1}{8}$th part in clothes, $\frac{1}{10}$th in incidental expenses, and saved $318 yearly. What was the amount of his yearly income?

Ans. 720.

19. A certain sum of money is to be divided among 3 persons, A, B, and C, as follows: A shall have $3000 less than the half of it, B $1000 less than the third part, and C is to receive $800 more than the fourth part of the whole sum. What is the sum to be divided? and what does each receive?

Ans. Whole sum $38,400.
A receives $16,200,
B receives $11,800,
C receives $10,400.

20. Says A to B, I have a certain number in my thoughts: if I multiply it by 7, add 3 to the product, divide this by 2, subtract 4 from the quotient, and 15 will remain. Now what is the number?

Ans. 5.

At this stage of the student's progress, he might be taught a different method of representing the unknown quantity. For instance,

(79.) *Divide a number* a *into two such parts that the first part shall be to the second as* m *to* n.

Now mx and nx will represent the numbers, whatever may be the value of x.

For $$mx : nx :: m : n.$$

If we, therefore, put mx=one part,

and nx=the other,

we shall have $$mx+nx=a,$$

or $$(m+n)x=a;$$

$$\therefore x=\frac{a}{m+n},$$

and one is $\frac{ma}{m+n}$, and the other $\frac{na}{m+n}$.

21. The sum of \$1200 is to be divided between two persons, A and B, so that A's share is to B's as 2 to 7. How much does each receive?

Ans. A receives \$266⅔.
B receives \$933⅓.

22. Divide the number 95 into two parts which shall be to each other as 8 to 11.

Ans. 40 and 55.

23. Two men 150 miles apart set out to meet each other; one goes 3 miles in the time the other goes 7. What part of the distance does each travel?

Ans. 45 and 105 miles.

24. A and B began to play; A with four ninths of the sum that B had. After A had won \$10, he had just the same sum that B had left. What had each at first? *Ans.* A \$16, and B \$36.

25. A person distributed £5 14*s*. among some poor women and children, giving to each woman 6 shillings, and to each child 2 shillings. The number of women was to the number of children as 4 to 7. How many were there of each?

Ans. 12 women and 21 children.

26. The number of days that 4 workmen were employed were severally as the numbers 4, 5, 6, 7; their daily wages being 3 shillings each; the sum received by the first and second was 36 shillings less than that received by the third and fourth. How much did each receive?

Ans. 36, 45, 54, and 63 shillings.

27. A purse of \$5700 is to be divided among three persons, A, B, and C; A's share is to be to B's as 6 to 11, and C is to have \$600 more than A and B together. What is the share of each?

Ans. A's \$900, B's \$1650, C's \$3150.

28. There are two numbers in proportion of 2 to 3, and if 4 be added to each of them, the sums will be in proportion of 5 to 7. What are the numbers?

Ans. 16 and 24

29. Divide \$315 among four persons, A, B, C, and D, giving B once and a half as much as A, C one third more than A and B together, and D one fourth more than A, B, and C. What is the share of each?

Ans. A \$24, B \$36, C \$80, D \$175.

30. Divide 36 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$d of the second, and $\frac{1}{4}$th of the third shall be equal to each other.

Ans. 8, 12, and 16.

31. A workman was employed for 60 days, on con-

dition that for every day he worked he should receive 15 pence, and for every day he was idle he should forfeit 5 pence; at the end of the time he had 20 shillings to receive. How many days did he work, and how many days was he idle?

Ans. 27 days worked,
33 days idle.

32. A person, after spending $100 more than one fifth of his income, had remaining $35 more than one half of it. What was his income?

Ans. $450.

33. A farmer had two flocks of sheep, each containing the same number. Having sold from one of them 39, and from the other 93, he finds twice as many remaining in the one as in the other. How many did each flock contain at first?

Ans. 147.

34. A shepherd being asked how many sheep he had, answered, If I had as many more, half as many more, and $7\frac{1}{2}$ sheep, I would have 500. How many had he?

Ans. 197.

35. A cistern can be filled by three pipes; by the first in 80 minutes, by the second in 200 minutes, and by the third in 300 minutes. In what time will the cistern be filled when all three pipes are open at once?

Ans. In 48 minutes.

36. Two gentlemen play at billiards; A, before he began to play, had $42, and B $24. Each lost and won in turn, when A found he had five times as much as B had remaining. How much did A win?

Ans. $13.

37. What capital is that which, with 5 years interest at 4 per cent., will amount to \$8208?

Ans. \$6840.

38. A capital was put out for one year at $4\frac{1}{2}$ per cent. *per annum;* at the expiration of the year there was received back, as capital and interest, \$13,167 What was the amount of the capital?

Ans. \$12,600.

39. A fortress has a garrison of 2600 men, among whom are nine times as many foot soldiers, and three times as many artillery as cavalry. How many are there in each *corps?*

Ans. 200 cavalry, 600 artillery, and 1800 foot.

40. Divide the number 46 into two parts, so that when the one is divided by 7, and the other by 3, the quotients together may amount to 10. What are the parts?

Ans. 28 and 18.

41. From the first of two mortars in a battery 36 shells are thrown before the second is ready for firing. Shells are then thrown from both in the proportion of 8 from the first to 7 of the second; the second mortar requiring as much powder for 3 charges as the first does for 4. It is required to determine after how many discharges of the second mortar the quantity of powder consumed by it is equal to the quantity consumed by the first.

(80.) Of Equations of the First Degree involving two or more Unknown Quantities.

We have before observed that an equation is the algebraic enunciation of a particular problem (*Art.* 68).

But a problem generally comprehends several different conditions independent of one another; these different conditions, being expressed in algebraic language, furnish a certain number of equations.

Thus, for example, let us propose

To find two numbers such that three times the first added to the second gives 23, *and that five times the first added to three times the second gives* 45.

We here find two propositions, each of which expresses a fact in two different terms.

1st. *The triple of an unknown quantity added to another unknown number, and then the equivalent*, 23.

2d. *Five times the first unknown number increased by three times the second, then the equivalent*, 45.

These can easily be translated into algebraic language. If we put x for the first, and y for the second unknown quantity, they become

$$3x+y=23, \text{ and } 5x+3y=45,$$

two equations *independent* of each other.

When two or more equations involving as many unknown quantities are independent of one another, they are called *determinate.*

Had we proposed the question,

Find two numbers such that three times the first added to the second gives 23, *and six times the first added to twice the second gives* 46. This second clause would express nothing more than the first, since we have only doubled two equivalents. We have therefore only one translation, and, consequently, but one equation. The question would then be *indeterminate.*

In order that a problem may be strictly limited,

there must be as many independent equations as there are unknown quantities.

To explain the methods of resolving problems of two or more unknown quantities, we shall take one of those which have already been solved by means of one unknown quantity only.

Given the sum of two numbers equal to s, and their difference equal to d, to find the numbers.

Let $x=$the greater,
and $y=$the less;
then $x+y=s$,
and $x-y=d$,
by addition, $2x=s+d$,
by subtraction, $2y=s-d$.

We first eliminated y, and then x; we have therefore two equations, each containing but one unknown quantity.

From the first we have $x=\frac{s+d}{2}$,

and from the second, $y=\frac{s-d}{2}$,

the same as before.

Elimination.

(81.) Elimination is the method of combining two or more equations containing as many unknown quantities, in such a manner as to deduce but one equation containing but one unknown quantity.

There are four principal methods of elimination:

1st. By addition or subtraction.

2d. By substitution.

3d. By comparison.

4th. By an indeterminate multiplier.

We shall explain these methods separately.

(82.) *Elimination by addition or subtraction.*

In order to simplify the calculations and avoid the inconvenience arising from the multitude of letters which must be employed to represent the given quantities, we shall distinguish by the same letter all the coefficients of the same unknown quantity; but we shall affect them with one or more accents, as a', a'', a''', &c., which are read, *a prime*, *a second*, *a third*, &c.

Any two simple equations, each involving the same two unknown quantities, can always be brought to the form

$$ax+by=c \quad . \quad . \quad . \quad . \quad (1)$$
$$a'x+b'y=c' \quad . \quad . \quad . \quad . \quad (2)$$

Where a, a', b, b', c, c' may be any numbers whatever, either whole or fractional, positive or negative.

If now a were equal to a', or b equal to b', we might, by a simple subtraction, form a new equation that would contain but one unknown quantity; the value of which might then be readily obtained.

If we now multiply the first equation by a' and the second by a, we shall obtain

$$aa'x+a'by=a'c \quad . \quad . \quad . \quad (3)$$
$$aa'x+ab'y=ac' \quad . \quad . \quad . \quad (4)$$

The coefficients of x are now equal, and subtracting the (4) from (3), we have

$$a'by-ab'y=a'c-ac',$$

or

$$(a'b-ab')y=a'c-ac';$$

$$\therefore y=\frac{a'c-ac'}{a'b-ab'}.$$

Or, if we multiply the (1) by b' and (2) by b, we shall have

$$ab'x+bb'y=b'c \quad . \quad . \quad . \quad (5)$$

$$a'bx+bb'y=bc' \quad . \quad . \quad . \quad (6)$$

and subtracting (5) from (6), we have

$$(a'b-ab')x=bc'-b'c$$

$$x=\frac{bc'-b'c}{a'b-ab'}.$$

The reason why this method is called the *method by addition* or *subtraction* is that the unknown quantities are canceled by addition or subtraction.

From the above solution we derive the following general

RULE.

Multiply the equations by the least numbers that will make the coefficients of the unknown quantity to be eliminated equal to each other.

Then, if the sign of that quantity is the same in both equations, subtract; but if different, add.

EXAMPLES.

1. Given $\qquad \frac{x}{2}-12=\frac{y}{4}+8,$

and $\qquad \frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27,$

to find the values of x and y.

By transposition, we have, from the first equation,

$$\frac{x}{2}-\frac{y}{4}=20 \quad . \quad . \quad . \quad . \quad (1)$$

and from the second,

$$\frac{x+y}{5}-\frac{2y-x}{4}+\frac{x}{3}=35 \quad . \quad . \quad (2)$$

Clearing these equations of fractions, and reducing, we have

$$2x-y=80 \quad . \quad . \quad . \quad . \quad (3)$$

and

$$47x-18y=2100 \quad . \quad . \quad . \quad (4)$$

If we now multiply the third equation by 18, there will result

$$36x-18y=1440 \quad . \quad . \quad . \quad (5)$$

The signs of the unknown quantity whose coefficients are equal, are alike; we must therefore subtract.

Hence $$11x=660;$$

$$\therefore\ x=60,$$

and

$$y=2x-80$$
$$=120-80$$
$$=40.$$

Verification:

$$\therefore\ \frac{60}{2}-12=\frac{40}{4}+8, \text{ first equation,}$$

$$18=18.$$

$$\frac{60+40}{5}+\frac{60}{3}-8=\frac{80-60}{4}+27, \text{ second equation,}$$

$$20+20-8=5+27$$

$$32=32.$$

2. Let us now take an example of three equations involving three unknown quantities.

Given

$$2x+4y-3z=22,$$
$$4x-2y+5z=18,$$
$$6x+7y-\ z=63,$$

to find the values of x, y, and z.

From the first equation, we find

$$x=\frac{22-4y+3z}{2}.$$

If we substitute this value in the second and third equations, we shall have

$$44-\ 8y+6z-2y+5z=18,$$

and $$66-12y+9z+7y-\ z=63.$$

Reducing, $$11z-10y=-26,$$

and $$8z-\ 5y=3.$$

From this last equation, we have

$$y=\frac{8z+3}{5}.$$

Substituting this value of y in the preceding equation, we obtain

$$11z-(16z+6)=-26,$$

or $$11z-\ 16z-6=-26$$

$$-5z=-20$$

$$z=4.$$

Hence $$y=7, \text{ and } x=3.$$

This method is called elimination by substitution, and may be enunciated as follows:

(83.) *Find the value of one of the unknown quantities in terms of the others, and substitute this value in the other equations. We shall then have one equation less, and continue the same operation until there is but one equation.*

3. Let us again take the equations

$$ax+by=c \quad . \quad . \quad . \quad . \quad (1)$$

$$a'x+b'y=c' \quad . \quad . \quad . \quad . \quad (2)$$

From the first we have

$$x=\frac{c-by}{a},$$

and from the second,

$$x=\frac{c'-b'y}{a'}.$$

Equate these two values of x, we obtain

$$\frac{c-by}{a}=\frac{c'-b'y}{a'};$$

hence $$a'c-a'by=ac'-ab'y$$

$$ab'y-a'by=ac'-a'c,$$

or $$(ab'-a'b)y=ac'-a'c;$$

$$\therefore\ y=\frac{ac'-a'c}{ab'-a'b}.$$

This is called *elimination by comparison*, and may be expressed thus,

(84.) *Find the value of the same unknown quantity in each of the equations, and equate those values two by two.*

Let, as before, the equations be

$$ax+by=c$$
$$a'x+b'y=c'.$$

Multiplying the first equation by some indeterminate quantity m, it will become

$$amx+bmy=mc,$$

and subtracting from this result the second equation, we have

$$(am-a')x+(bm-b')y=cm-c'.$$

And, since the value of m is indeterminate, we can take

$$am-a'=0,$$

or $$m=\frac{a'}{a},$$

in which case the first term will disappear, and we shall have

$$\left(\frac{a'b}{a}-b'\right)y=\frac{a'c}{a}-c',$$

or $$(a'b-ab')y=a'c-ac';$$

$$\therefore\ y=\frac{a'c-ac'}{a'b-ab'},$$

which is the same value of y as before.

We might have made

$$bm-b'=0,$$

or

$$m=\frac{b'}{b}.$$

From which we would have found the value of x to be

$$x=\frac{cb'-c'b}{ab'-a'b};$$

or, by changing the signs of the numerator and denominator,

$$x=\frac{c'b-cb'}{a'b-ab'}.$$

This method, given by *Bezout*, is very simple, and possesses the advantage of deducing the values of each of the unknown quantities from the same equation.

Analysts make use of various other methods of elimination, but the above are most generally in use

To apply them to any number of equations m, we may have the following

Rule.

(85.) *To eliminate one of the unknown quantities, combine any one of the equations with each of the* m$-$1 *others; we shall then have* m$-$1 *new equations containing* m$-$1 *unknown quantities. To eliminate another, combine any one of the* m$-$1 *equations with each of the* m$-$2 *others; we shall then have* m$-$2 *new equations containing* m$-$2 *unknown quantities.*

Continue this series of operations until there is but one equation containing one unknown quantity; the value of which may then be obtained, and thence all the rest.

EXAMPLES.

1. Given $11x+3y=100$, and $4x-7y=4$, to find the values of x and y.

Ans. $x=8$, $y=4$.

2. Given $x+2y=17$, and $3x-y=2$, to find the values of x and y.

Ans. $x=3$, $y=7$.

3. Given $3x+5y=40$, and $x+2y=14$, to find the values of x and y.

Ans. $x=10$, $y=2$.

4. Given $\frac{x+8}{4}+6y=21$, and $\frac{y+6}{3}=23-5x$, to find x and y.

Ans. $x=4$, $y=3$.

5. Given $$\frac{3x-5y}{2}=\frac{2x+y}{5}-3,$$

and $$8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3},$$

to find x and y.

Ans. $x=12$, $y=6$.

6. Given $$3x+\frac{7y}{2}=22,$$

and $$11y-\frac{2x}{5}=20,$$

to find x and y.

Ans. $x=5$, $y=2$.

7. Given $5+\frac{2x+7-y}{15}=\frac{3x+4y+3}{10}-\frac{y-8}{5}$,

and $\frac{7x+6}{11}=\frac{9y+5x-8}{12}-\frac{x+y}{4}$,

to find the values of x and y.

Ans. $x=7$, $y=9$.

8. Given $\frac{2x}{3}-4+\frac{y}{2}+x=8-\frac{3y}{4}+\frac{1}{12}$,

and $\frac{1}{6}-2x+6=\frac{y}{6}-\frac{x}{2}+2$,

to find the values of x and y.

Ans. $x=2$, $y=7$.

9. Given $\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}$,

and $\frac{2y+4}{3}-\frac{2x+y}{8}=\frac{x+13}{4}$,

to find x and y.

Ans. $x=7$, $y=10$.

10. Given $\frac{25+5y}{6}-\frac{7x-6}{3}=9-\frac{3x-10+7y}{12}$,

and $\frac{96-8x}{9}=5x-\frac{14+y}{3}$,

to find the values of x and y.

Ans. $x=3$, $y=7$.

11. Given $x-\frac{3x+5y}{17}+17=5y+\frac{4x+7}{3}$,

and $\frac{22-6y}{3}-\frac{5x-7}{11}=\frac{x+1}{6}-\frac{8y+5}{18}$,

to find the values of x and y.

Ans. $x=8$, $y=2$.

12. Given $2x+4y-3z=22$,

$$4x-2y+5z=68,$$
$$5x+\ y-2z=14,$$

to find x, y, and z.

Ans. $x=3$, $y=7$, $z=4$.

13. Given $x+\ y+\ z=29$,

$$x+2y+3z=62,$$
$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10,$$

to find the values of x, y, and z.

Ans. $x=8$, $y=9$, $z=12$.

14. Given $\frac{1}{x}+\frac{1}{y}=\frac{1}{8}$ . . . . (1)

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{9} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{10} \quad . \ . \ . \ . \ . \ . \ . \quad (3)$$

to find the values of x, y, and z.

If we subtract the second from the first, we have

$$\frac{1}{y}-\frac{1}{z}=\frac{1}{72} \quad . \ . \ . \ . \quad (4)$$

Add (3) and (4), there will be

$$\frac{2}{y}=\frac{41}{360}$$
$$41y=720$$
$$y=17\tfrac{23}{41}.$$

If we subtract (4) from (3), we shall have

$$\frac{2}{z}=\frac{31}{360}$$
$$31z=720$$
$$z=23\tfrac{7}{31}.$$

Substitute the value of y in the first equation, we have

$$\frac{1}{x}+\frac{41}{720}=\frac{1}{8}$$

$$720+41x=90x$$

$$49x=720$$

$$x=\frac{720}{49}=14\tfrac{34}{49}.$$

Second Solution.—If we add the three equations together, we shall have

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{121}{360};$$

or, by dividing by 2, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{121}{720}$.

Subtracting each equation from this in succession, there will result

$$\frac{1}{z}=\frac{31}{720} \quad \therefore \quad z=23\tfrac{7}{31}$$

$$\frac{1}{y}=\frac{41}{720} \quad \therefore \quad y=17\tfrac{23}{41}$$

$$\frac{1}{x}=\frac{49}{720} \quad \therefore \quad x=14\tfrac{34}{49}.$$

(86.) Problems producing Equations of the First Degree involving more than one Unknown Quantity.

The usual method of solving problems of the first degree, is to assume as many unknown letters as there are unknown numbers to be found; then, having properly examined the conditions of the problem, translate them into as many distinct equations; then, by the resolution of these equations, according to the rules

just passed over, the quantities sought will be determined.

Examples.

1. A banker has two kinds of money; it takes a pieces of the first to make a dollar, and b pieces of the second to make the same sum. Some one offers him a dollar for c pieces. How many of each kind must the banker give him?

Let x=the number of pieces of the 1st kind, and
y= " " " 2d "

Then, because it takes a pieces to make a dollar, one piece will be the $\frac{1}{a}$th part of a dollar, and x pieces will therefore be the $\frac{x}{a}$*th* part of a dollar. For a similar reason, y pieces will be the $\frac{y}{b}$*th* part of a dollar; and these two fractions of a dollar make a whole dollar; therefore,

$$\frac{x}{a}+\frac{y}{b}=1,$$

$$x+y=c.$$

Clearing the first equation of fractions, it becomes

$$bx+ay=ab.$$

If we multiply the second equation by a, we shall eliminate y; or, if we multiply it by b, we shall eliminate x. The first operation gives us

$$bx+ay=ab$$

$$ax+ay=ac;$$

$$\therefore ax-bx=ac-ab,$$

or $$x=\frac{ac-ab}{a-b}=\frac{a(c-b)}{a-b}.$$

The second operation gives us

$$bx+ay=ab$$
$$bx+by=bc\,;$$
$$\therefore\ ax-by=ab-bc,$$

or $$(a-b)y=(a-c)b\,;$$

$$\therefore\ y=\frac{b(a-c)}{a-b}.$$

If $a=16$, $b=8$, and $c=10$, we shall have

$$x=\frac{16(10-8)}{16-8}=\frac{32}{8}=4$$
$$y=\frac{8(16-10)}{16-8}=\frac{48}{8}=6\,;$$

that is, 6 shilling pieces and 4 sixpenny pieces.

Verification, 6 shillings $=\frac{3}{4}$th of a dollar,

4 sixpences $=\frac{1}{4}$th of a dollar,

$\frac{3}{4}+\frac{1}{4}=1$, and $6+4=10$.

2. Find what each of three persons, A, B, and C, is worth, knowing, 1st. That what A is worth, added to l times what B and C are worth, is equal to p. 2d. That what B is worth, added to m times what A and C are worth, is equal to q. 3d. That what C is worth, added to n times what A and B are worth, is equal to r.

Let $x=$ what A is worth,

$y=$ " B "

$z=$ " C "

then, by the question,

$$x+l(y+z)=p \quad . \quad . \quad . \quad (1)$$
$$y+m(x+z)=q \quad . \quad . \quad . \quad (2)$$
$$z+n(x+y)=r \quad . \quad . \quad . \quad (3)$$

If to and from the first member of (1) we add and subtract lx, and to and from the first member of (2) we add and subtract my, and to and from the first member of (3) we add and subtract nz, these equations will become

$$(1-l)x+l(x+y+z)=p \quad . \quad . \quad . \quad (4)$$

$$(1-m)y+m(x+y+z)=q \quad . \quad . \quad . \quad (5)$$

$$(1-n)z+n(x+y+z)=r \quad . \quad . \quad . \quad (6)$$

From (4) $\quad x+\frac{l}{1-l}(x+y+z)=\frac{p}{1-l} \quad . \quad . \quad . \quad (7)$

From (5) $\quad y+\frac{m}{1-m}(x+y+z)=\frac{q}{1-m} \quad . \quad . \quad . \quad (8)$

From (6) $\quad z+\frac{n}{1-n}(x+y+z)=\frac{r}{1-n} \quad . \quad . \quad . \quad (9)$

Adding (7), (8), and (9) together, we have

$$x+y+z+\left(\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}\right)(x+y+z)=\frac{p}{1-l}+\frac{q}{1-m}$$
$$+\frac{r}{1-n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or $\left\{1+\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}\right\}(x+y+z)=\frac{p}{1-l}+\frac{q}{1-m}$

$$+\frac{r}{1-n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\therefore\ x+y+z=\frac{\frac{p}{1-l}+\frac{q}{1-m}+\frac{r}{1-n}}{1+\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}} \quad . \quad (12)$$

Substitute this value of $x+y+z$ in equations (7), (8), (9), and we have

$$x=\frac{p}{1-l}-\frac{l}{1-l}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\}. \quad (13)$$

$$y=\frac{q}{1-m}-\frac{m}{1-m}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\}. \quad (14)$$

$$z=\frac{r}{1-n}-\frac{n}{1-n}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\}. \quad (15)$$

3. There is a certain number, consisting of two figures, the sum of those figures is 5; and if 9 be added to the number itself, the digits will be inverted. What is the number?

Ans. 23.

4. What fraction is that to the numerator of which if 4 be added, the value is one half, but if 7 be added to the denominator, its value is one fifth?

Ans. $\frac{5}{18}$.

5. A person has two horses and two saddles, one of which cost \$50, the other \$2. If he places the better saddle upon the first horse, and the worse upon the second, then the latter is worth \$8 less than the other; but if he puts the worse saddle upon the first horse, and the better upon the other, then the latter is worth $\frac{15}{4}$ times as much as the first. What is the value of each horse?

Ans. The first \$30, the second \$70.

6. What fraction is that whose numerator being

doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$d; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$th?

Ans. $\frac{4}{5}$.

7. If A and B together can perform a piece of work in 12 days, B and C together in 20 days, and A and C in 15 days, how many days would it take each person to perform the same work alone?

Ans. A 20 days,
B 30 days,
C 60 days.

8. A cistern can be filled by three pipes, A, B, and C. The pipes A and B can fill it in 8 minutes; A and C together in 9 minutes; and B and C in 10 minutes. How many minutes would it take each pipe to fill it alone?

Ans. A $14\frac{34}{49}$ minutes,
B $17\frac{23}{41}$ minutes,
C $23\frac{7}{31}$ minutes.

9. A person bought 9 horses and 7 cows for \$300; and afterward, at the same prices, he bought 6 horses and 13 cows for the same sum. What was the price of each?

Ans. \$24 price of a horse,
\$12 price of a cow.

10. Find three numbers such that the first, with half the sum of the second and third, shall be 120; the second, with one fifth the difference of the third and first, shall be 70; and half the sum of the three numbers shall be 95.

Ans. 50, 65, and 75.

11. In 4000 pounds of gunpowder there are 3240

pounds less of sulphur than of charcoal and saltpetre, 2760 pounds less of charcoal than of sulphur and saltpetre. How many pounds of each?

Ans. 380 pounds sulphur,
620 " charcoal,
3000 " saltpetre.

12. The two hands of a clock are together at 12. At what times will they be together during the next 12 hours?

Ans. $\overset{\text{h.}}{1}\ \overset{\text{m.}}{5}\tfrac{5}{11}$, $\overset{\text{h.}}{2}\ \overset{\text{m}}{10}\tfrac{10}{11}$, &c.

13. A number is expressed by three figures; the sum of these figures is 8; the figure in the place of units is five times that in the place of hundreds, and when 396 is added to the number, the sum obtained is expressed by the figures of this number reversed. What is the number?

Ans. 125.

14. Two persons have the same income; A saves one fifth of his yearly; but B, by spending £50 per annum more than A, at the end of 4 years finds himself £100 in debt. What is the income?

Ans. £125.

15. Divide the number 10 into three such parts that if the first be multiplied by 2, the second by 3, and the third by 4, the products shall be equal.

Ans. $4\tfrac{8}{13}$, $3\tfrac{1}{13}$, and $2\tfrac{4}{13}$.

16. If A gives B 5 shillings of his money, B will have twice as much as A has left; but if B gives A 5 shillings of his money, A will have three times as much as B has left. How much has each?

Ans. A 13 shillings, B 11 shillings.

17. A man and his wife usually drank out a cask

of beer in 12 days, but when the man was from home, it lasted the woman 30 days. How many days would the man be in drinking it alone?

Ans. 20 days.

18. There are three numbers such that the second exceeds the first as much as the third exceeds the second; and the first is to the third as 5 to 7; also, the sum of the three numbers is 324. What are the numbers? *Ans.* 90, 108, and 126.

19. What two numbers are those whose difference, sum, and product are to each other as the numbers 2, 3, and 5 respectively?

Ans. 10 and 2.

20. Divide the number 90 into four such parts that if the first be increased by 2, the second diminished 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient shall all be equal. *Ans.* 18, 22, 10, and 40.

21. Divide the number 36 into three such parts that $\frac{1}{2}$ the first, $\frac{1}{3}$d of the second, $\frac{1}{4}$th of the third shall all be equal to each other.

Ans. The parts are 8, 12, and 16.

22. When a company came to pay their bill, they found that if there had been three persons more, they would have had a shilling a piece less to pay, and if there had been two less, they would have had a shilling a piece more to pay. Required the number of persons, and what each had to pay.

Ans. 12 persons, and each had 5 shillings to pay.

23. A person left $28,000 between his son, daughter, and niece; he ordered the son to have $4 as often as the other two had $3, and the daughter $4 as often

as the other two should have $10. What were the shares of each?

Ans. Son's share, $16,000;
Daughter's share, $8,000;
Niece's share, $4,000.

24. A person buys three loads of grain; the first, containing 30 bushels of rye, 20 of barley, and 10 of wheat, for 230 shillings; the second, containing 15 bushels of rye, 6 of barley, and 12 of wheat, for 138 shillings; the third, containing 10 bushels of rye, 5 of barley, and 4 of wheat, for 75 shillings. What was the price per bushel of each?

Ans. Rye, 4 shillings per bushel;
Barley, 3 " " "
Wheat, 5 " " "

25. A field of wheat and rye, which contained 20 acres, was put to a laborer to reap for 126 shillings; the wheat at 7 shillings per acre and the rye at 5 shillings. The laborer, falling ill, reaped only the rye. How much money ought he to receive?

Ans. 35 shillings.

26. A person has three horses and a saddle, which of itself is worth $220; now, if the saddle be put on the back of the first horse, it will make his value equal to that of the second and third; but if it be put on the back of the second, it will make his value double that of the first and third; and if it be put on the back of the third, it will make his value triple that of the first and second. What is the value of each horse?

Ans. 1st horse =$20,
2d " =$100,
3d " =$140.

(87.) Discussion of Equations of the First Degree.

Every equation of the first degree may be reduced to the form

$$ax+b=cx+d,$$

or, by transposition, $ax-cx=d-b$,

$$\therefore \quad x=\frac{d-b}{a-c}.$$

In the solution of this equation three different cases may present themselves.

1. $d>b$ and $a>c$.
2. $d<b$ and $a>c$, or $d>b$ and $a<c$.
3. $d<b$ and $a<c$.

The first case offers no difficulty; in the second and third cases, however, the interpretation is not so clear. We shall therefore proceed to examine them.

In the second case it is evident, if $d<b$, and $a>c$, the equation

$$ax+b=cx+d$$

is absurd, for ax and b in the first member are respectively greater than the two terms cx and d in the second member. We shall then perceive that the problem, of which the equation is a translation, is inconsistent.

In the third case, we suppose that $d<b$ and $a<c$. Now, as we can not in an arithmetical sense subtract a greater quantity from a less, let us further examine our first equation. We see that, instead of having

$$ax-cx=d-b,$$

we might with equal propriety have had

$$cx-ax=b-d,$$

or

$$x=\frac{b-d}{c-a}.$$

There is no obscurity now in the value of x. All we have to do when a difficulty of this kind presents itself is to change the signs of the terms in both numerator and denominator.

In the second case, we might have substituted $-x$ for $+x$, and we would have had

$$-ax+b=-cx+d$$
$$ax-cx=b-d,$$
$$\therefore\ x=\frac{b-d}{a-c}.$$

If we then modify the question in such a manner as to correspond with this new equation, it will no longer be absurd. For instance,

A father has lived 45 years and his son 15; in how many years will the father be four times as old as his son?

Let x = the number of years required;
then $45+x$ = the father's age at that time,
and $15+x$ = the son's age;

$$\therefore\ 60+4x=45+x,$$
or
$$3x=-15$$
$$x=-5.$$

Verification, $60+(-20)=45+(-5)$,
or
$$60-20=45-5$$
$$40=40.$$

We perceive that the value of x then *verifies* the equation; but how shall we interpret it? If we substitute $-x$ for $+x$, as we had a right to do, we should have had

$$60-4x=45-x$$
$$3x=15$$
$$x=5.$$

This changes the conditions of the question to the following: A father has lived 45 years and his son 15; how many years have elapsed since the father was four times as old as his son?

A negative result then shows some incompatibility in the enunciation of the question. The language of Algebra detects the error, and points out the rectification of the mistake. We may from this establish the following general principles.

1°. *When we find a negative value for the unknown quantity in problems of the first degree, it points out an absurdity in the conditions of the problem; but*

The value so obtained, with a positive sign, may be considered as the answer to a problem which differs from the one proposed in this only, that certain quantities which were additive in the first have become subtractive in the second, and reciprocally.

2°. There are cases in which the negative value has still a different interpretation; for instance, in determining the velocity with which a ball rebounds, v represents the velocity which the ball has before impact; and if we find the value of the returning velocity to be $=-v$, it shows that the two velocities are equal, but in contrary directions (*Art.* 3). As a further illustration of the properties of equations, we shall discuss the following problem:

Two pedestrians are traveling along the same highway, in the same direction, the hinder one goes m miles per hour, the forward one n miles per hour; at a certain time of day they are a miles apart. In what time will they be together?

Let t=the time required in hours.

Then, as the forward one goes n miles per hour, he will travel nt miles.

The hinder one will travel mt miles;

$$\therefore\ mt-nt=a,$$

or $$t=\frac{a}{m-n}.$$

1° Now if $m>n$, the value of t will be positive. The hinder one will eventually overtake the forward one, and the time must be added to the given time.

2°. If $m<n$, the value of t will be negative. The hinder one will never overtake the forward one; but as they have been traveling along the same road, there was a time when they were together, and this time is to be subtracted from the given time (*Art.* 3).

3°. If $m=n$, then $t=\frac{a}{0}$; that is, they are a miles apart, and both travel at the same rate, they will always be a miles apart; $\frac{a}{0}$ is therefore a symbol of infinity.

Resuming the expression $\frac{a}{0}$, we can put it equal to ∞ (*Art.* 10);

$$\therefore\ \frac{a}{0}=\infty.$$

Clearing this equation of fractions, we have

$$a=0\times\infty.$$

From which it appears that zero, multiplied by infinity, is a finite quantity; again, dividing both members of this equation by ∞, we have

$$\frac{a}{\infty}=0;$$

that is, a finite quantity being divided into an infinite number of equal parts, each of those parts is zero.

4°. If $a=0$, and $m=n$, then

$$t=\frac{0}{0};$$

that is, they are both together, and travel at the same rate; hence they will be together at *all times*. The symbol $\frac{0}{0}$ is then a symbol of indetermination.

The student is not at liberty to suppose, however, that the symbol $\frac{0}{0}$ is always indeterminate. It sometimes denotes the presence of a common factor, which being canceled, the true value may be obtained.

Suppose that $\qquad x=\dfrac{a^2-b^2}{a-b}.$

When $b=a$, the value of x becomes $\frac{0}{0}$; but

$$a^2-b^2=(a+b)(a-b);$$

$$\therefore\ x=\frac{(a+b)(a-b)}{a-b}$$

$$=a+b,$$

and when $b=a$, then $x=2a$.

5°. Suppose $a=0$, and m and n unequal, then

$$t=0;$$

that is, if we suppose the two pedestrians to be together, and travel at unequal rates, this will be the only point in which they can be together.

The solution of a problem containing equations of the first degree may give one of the four following results, viz., *positive*, *negative*, *infinite*, or *indeterminate.*

OF INEQUALITIES.

(88.) In the discussion of problems it is frequently necessary to find the limits of the unknown quantity. These limits are obtained by the method of *inequalities.*

An inequality consists of two members separated by $>$ or $<$. The principles explained in the solution of equations, for the most part, apply equally to those of inequalities. There are, however, some important exceptions which it is necessary to point out, in order that the student may not fall into error. We shall discuss the different transformations in succession.

1°. *If we add the same quantity to or subtract it from the two members of any inequality, the inequality will subsist in the same sense;* that is, if

$$a>b, \text{ then } a+m>b+m,$$

or

$$a-m>b-m.$$

Two inequalities subsist in the same sense, when the greater quantity is in the first member of both, or in the second member of both; and, in a contrary sense, when the greater quantity is in the first member of the one, and in the second member of the other.

2°. *If we add the corresponding members of two or more inequalities which subsist in the same sense, the resulting inequality will subsist in the same sense as they do.*

Thus, if
$$a>b,$$
$$c>d,$$
$$e>f,$$
then
$$a+c+e>b+d+f.$$

3°. *But if we subtract the corresponding members of two or more inequalities, subsisting in the same sense, the resulting inequality will not always subsist in the same sense as they do.*

For, take the inequalities

$5<8$, $3<4$, we shall have

$2<4$, which is correct;

but if we take $11<12$

$5<9$, we shall have

$11-5>$ not $<12-9$,

or $6>$ not <3.

4°. *If we multiply or divide both members of an inequality by a positive quantity, the resulting inequality will subsist in the same sense.*

$$a>b, \text{ then } ma>mb, \text{ or } \frac{a}{m}>\frac{b}{m},$$

$$-a>-b, \text{ then } -ma>-mb, \text{ or } \frac{-a}{m}>-\frac{b}{m}.$$

By this means we can clear an inequality of fractions.

5°. *But if we multiply or divide both members of an inequality by the same negative quantity, the resulting inequality will subsist in a contrary sense.*

Thus, $20>15$.

Now multiply both members by -4, we shall have

$$-80<-60.$$

Hence, if we change the signs of both members of

an inequality, we must also reverse the sign of the inequality itself.

6°. *If both members of an inequality be positive members, we can raise them to any power without changing the sense of the inequality.*

Thus, $a>b$,

then $a^n>b^n$.

7°. *We can also extract the root of both members of an inequality without altering the sense of the inequality.*

EXAMPLES.

1. Find the limit of the value of x in the inequalities

$$\frac{ax}{5}+bx-ab>\frac{a^2}{5} \quad . \quad . \quad . \quad (1)$$

$$\frac{bx}{7}-ax+ab<\frac{b^2}{7} \quad . \quad . \quad . \quad (2)$$

Clearing the inequalities of fractions, we have

$$ax+5bx-5ab>a^2 \quad . \quad . \quad . \quad (3)$$

$$bx-7ax+7ab<b^2 \quad . \quad . \quad . \quad (4)$$

From (3), by adding $5ab$ to both members, and factoring,

$$(a+5b)x>(a+5b)a;$$

$$\therefore\ x>a.$$

From (4), $(b-7a)x<(b-7a)b;$

$$\therefore\ x<b,$$

a result which can be verified. In verifying an inequality, if we take x=the limit, the inequality will become an equation.

2. Find the limit of the value of x in the expression

$$x+\tfrac{1}{2}x+\tfrac{1}{3}x>11.$$

Ans. $x>6$.

3. Find the limit of the value of x in the expression

$$\tfrac{1}{2}x+3x-5>16.$$

Ans. $x>6$.

4. Find the limit of the value of x in the expression

$$\tfrac{1}{2}x+\tfrac{1}{3}x+\tfrac{1}{4}x+\frac{x}{6}+\frac{x}{12}-7>9.$$

Ans. $x>12$.

5. Given $$\frac{bx}{2}+cx-ac<\frac{ab}{2},$$

and $$cx+\frac{dx-bd}{3}>bc,$$

to find the limits of x.

Ans. $x<a$ and $x>b$.

6. Given $3x+7x-30>10$ to find the limit of x.

Ans. $x>4$.

7. The double of a number, diminished by 6, is greater than 24; and triple the number, diminished by 6, is less than double the number increased by 10. Required a number which will fulfill the conditions.

Ans. $x>15$ and $x<16$.

8. What number is that whose half and third part added together are less than 105, but its half, diminished by its fifth part, is greater than 33?

Ans. Greater than 110, and less than 126.

CHAPTER IV.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

(89.) The *square root* of a number is such a number as, being multiplied by itself, will produce the proposed number.

Every number may be regarded as composed of a certain number of units, plus a certain number of tens, plus a certain number of hundreds, &c. Thus, 27 contains 7 units and 2 tens; or, if we represent the tens by a and the units by b, we shall have

$$27=2 \text{ tens } +7 \text{ units};$$

$$\therefore\ a+b=20+7.$$

If we now square both members of this equation, we shall have

$$(a+b)^2=(20+7)^2,$$

or

$$a^2+2ab+b^2=400+280+49$$
$$=729.$$

Hence *the square of a number composed of tens and units consists of the square of the tens, plus twice the product of the tens by the units, plus the square of the units.*

If we reverse this process, we shall find the square root of the number.

We perceive that the square of 10 is 100, the square of 100 is 10,000, and so on. Therefore the square root of any square number less than 100 consists of one figure, and of any square number over 100, and less

than 10,000, of two figures. Hence *every two figures in the power demands one figure in its root.*

Therefore, to find the number of figures in a root, we must *separate the given number into periods of two figures each, beginning at the units place. Then, to extract the square root, find the greatest square in the left-hand period, and place its root to the left, as the first figure of the root; subtract its square from the first period, and to the remainder bring down the next period for a dividend.*

Divide this new dividend (excluding the right-hand figure) by twice the first figure of the root, which place to the right of the first figure of the root and also to the right of the partial divisor.

Multiply the complete divisor by the last figure of the root, subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.

Double the whole root already found for a partial divisor, and continue the operation, as before, until all the periods are brought down. If the last remainder is zero, the proposed number is a perfect square.

If there be a remainder, we have extracted the square root of the greatest perfect square contained in the number, and have obtained the *entire part of the root.*

Let it be required to extract the square root of 288.

$$
\begin{array}{r|l}
 & 288\ (16 \\
 & 1 \\
\hline
26 & 188 \\
 & 156 \\
\hline
 & \ \ 32
\end{array}
$$

Here we have a remainder greater than the divisor. But to show that 16 is the entire part of the root, we must prove that *the difference of the squares of two consecutive numbers is equal to the sum of the numbers.* *Then,*

If the remainder is less than the sum of the root and its next consecutive number, we have obtained the integral part of the root.

Put a = the root,

and $a+1$ = the next consecutive number;

then
$$(a+1)^2-a^2=a^2+2a+1-a^2$$
$$=2a+1$$
$$=a+(a+1).$$

To extract the square root of a number which is not a perfect square to within a given fraction.

Let a = the number,

then a may be put under the form
$$\frac{an^2}{n^2}.$$

If we put r = the integral part of the root of the numerator an^2, the number an^2 will be comprised between r^2 and $(r+1)^2$, and
$$\frac{an^2}{n^2}<\frac{(r+1)^2}{n^2},$$
and
$$\frac{an^2}{n^2}>\frac{r^2}{n^2};$$
$$\therefore \sqrt{\frac{an^2}{n^2}}=\sqrt{a}<\frac{r+a}{n},$$
and
$$\sqrt{\frac{an^2}{n^2}}=\sqrt{a}>\frac{r}{n}.$$
But
$$\frac{r+1}{n}-\frac{r}{n}=\frac{1}{n}.$$

Hence, to extract the square root of a number not a perfect square to within a given fraction,

Multiply the given number by the square of the denominator of the fraction that marks the degree of approximation; extract the square root of the product to the nearest unit, and divide the quotient by the denominator of the fraction.

The method of extracting the square root of decimals is derived from this rule.

ON THE FORMATION OF POWERS.

(90.) *The powers of any quantity are the successive products arising from multiplying that quantity by itself. The square, or second power, is the product arising from multiplying a quantity by itself once; the* m*th power is the product arising from multiplying the quantity by itself* m−1 *times.*

Thus, the first power of a is $a^1=a$,
" second " a is $a\times a=a^2$,
" third " a is $a\times a\times a=a^3$,
" fourth " a is $a\times a\times a\times a=a^4$,
" mth " a is $a\times a\times a$ . . . . .

repeated as a factor m times, or a multiplied by itself $m-1$, and is equal to a^m.

If we multiply $\pm 4a^2b$ by itself once, we shall have

$$\pm 4a^2b\times \pm 4a^2b=16a^4b^2.$$

Hence, in order to square a monomial, *we must square its coefficient and multiply the exponent of each letter by* 2.

We also perceive that, *whatever may be the sign of a monomial, its square is positive.*

If we continue the multiplication of $\pm 4a^2b$, we shall have

$$\pm 4a^2b \times \pm 4a^2b \pm 4a^2b^3 = \pm 64a^6b^3$$
$$\pm 4a^2b \times \pm 4a^2b \pm 4a^2b \pm 4a^2b = 256a^8b^4.$$

An even power of a positive or negative quantity is always positive. For 2m *being the form of even numbers* (*Art.* 52), we shall have

$$(\pm a)^{2m} = \{(\pm a)^2\}^m = (+a^2)^m = +a^{2m}.$$

Any odd power of a quantity will have the same sign as the quantity itself.

For $2m+1$ being the form of odd numbers (*Art.* 52), we shall have

$$(\pm a)^{2m+1} = +a^{2m} \times \pm a = \pm a^{2m+1}.$$

CASE I.

(91.) *To raise a monomial to any power.*

RULE.

Raise the coefficient to the given power, and multiply the exponent of each letter by the index of the required power.

EXAMPLES.

1. Required the fifth power of $-2a^3x^2y$.

Here $(-2a^3x^2y)^5 = -2^5 . a^{15}x^{10}y^5 = -64a^{15}x^{10}y^5$, *Ans.*

2. Required the fourth power of $-\frac{b}{2a}$.

Here $$\left(-\frac{b}{2a}\right)^4 = \frac{b^4}{2^4a^4} = \frac{b^4}{16a^4}.$$

3. Required the square of $4ab^2c^3$.

Ans.

4. Required the third power of $5ax^4y^3$.

Ans.

5. Required the fourth power of $-6a^3x^3y$.

Ans.

6. Required the fifth power of $\dfrac{ax^2}{2bc}$.

Ans.

7. Required the sixth power of $\dfrac{abc}{xy^2z}$.

Ans.

8. Required the seventh power of $-bc^2d^3$.

Ans.

Expressions with *negative* exponents are subje
the same rules as those with *positive* exponents.

Thus, required the fifth power of a^{-4}.

Here $(a^{-4})^5=a^{-20}$, or, by (*Art.* 43),

$$a^{-4}=\frac{1}{a^4} \therefore (a^{-4})^5=\left(\frac{1}{a^4}\right)^5=\frac{1}{a^{20}}.$$

9. Required the mth power of a^nb.

$$(a^nb)^m=a^{mn}b^m.$$

10. Required the third power of $3a^2b^{-3}$.

Ans.

11. Required the seventh power of $a^2b^{-3}c^{-3}$.

Ans.

12. Required the zero power of xy.

Ans.

CASE II.

(92.) *To raise a polynomial to any power.*

RULE.

Multiply the polynomial by itself as many t
less one as is denominated by the index of the po

EXAMPLES.

1. Thus, what is the fifth power of $a+b$? we have

$$
\begin{array}{l}
a+b \\
\underline{a+b} \\
a^2+ab \\
\underline{\quad +ab+b^2} \\
(a+b)^2=a^2+2ab+b^2, \text{ the 2d power of } (a+b). \\
\underline{a+b} \\
a^3+2a^2b+ab^2 \\
\underline{\quad +a^2b+2ab^2+b^3} \\
(a+b)^3=a^3+3a^2b+3ab^2+b^3, \text{ the 3d power of } (a+b). \\
\underline{a+b} \\
a^4+3a^3b+3a^2b^2+ab^3 \\
\underline{\quad +a^3b+3a^2b^2+3ab^3+b^4} \\
(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4, \text{ the 4th power.} \\
\underline{a+b} \\
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\underline{\quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5} \\
(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5, \text{ the 5th power.}
\end{array}
$$

2. Required the square of $a+2b$.

Ans. $a^2+4ab+4b^2$.

3. What is the cube of a^2-x^2?

Ans. $a^6-3a^4x^2+3a^2x^4-x^6$.

4 What is the fourth power of $a+3b$?

Ans. $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$.

5. What is the square of $3x^2+2x+5$?

Ans. $9x^4+12x^3+34x^2+20x+25$.

6. What is the cube of $3x-5$?

Ans. $27x^3-135x^2+225x-125$.

7. Required the cube of x^2-2x+1.

Ans. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

8. Required the square of $a+b+c+d$.

Ans. $a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$.

From Example 8 we infer that *the square of any polynomial is composed of the sum of the squares of all the terms, together with twice the sum of the products of all the terms multiplied together two and two.*

(93.) Extraction of the Square Root of Monomials, and the Calculus of Radicals of the Second Degree.

The square root of any algebraic quantity is that quantity which, being multiplied by itself, will produce the proposed quantity.

We have seen (*Art.* 90) that, in order to square a monomial, we must square its coefficient, and multiply the exponent of each letter by 2. Hence, in order to extract the square root of a monomial, we must

Extract the square root of its coefficient, divide the exponent of each letter by 2, and prefix the double sign of plus and minus.

As a fraction is squared by squaring the numerator and denominator separately, therefore, to extract the square root of a fraction which is a perfect square,

Extract the square root of its numerator and denominator separately.

A monomial is a perfect square when its coefficient is a perfect square, and the exponent of each letter is an even number.

An even root of a negative quantity is an imagin-

ary expression. Such quantities may be represented by the form $\sqrt{a}\sqrt{-1}$.

When an *imperfect square* is introduced, it is affected with the radical sign $\sqrt{}$. Such an expression is called *a radical of the second degree. It is the indicated root of an imperfect square;* and, although its root can not be obtained exactly, still the expression may frequently be simplified. For we have

$$\sqrt{x}\times\sqrt{x}=(\sqrt{x})^2=x$$

$$\sqrt{xy}\times\sqrt{xy}=(\sqrt{xy})^2=xy, \text{ and } \sqrt{a}\times\sqrt{b}=\sqrt{ab}$$

$$\sqrt{a}\times\sqrt{b}\times\sqrt{c}\times\sqrt{d}=\sqrt{abcd}$$

$$(\sqrt{a}\times\sqrt{b}\times\sqrt{c}\times\sqrt{d})^2=(\sqrt{abcd})^2=abcd.$$

Hence, *the product of the square roots of any number of factors is equal to the square root of the product of those factors.*

We may, therefore, write

$$\sqrt{16a^2x}=\sqrt{16a^2\times x}=\sqrt{16a^2}\times\sqrt{x},$$

but $$\sqrt{16a^2}=\pm 4a;$$

$$\therefore\ \sqrt{16a^2x}=\pm 4a\sqrt{x}.$$

Again, $$5\sqrt{192a^3b^4c}=5\sqrt{64a^2b^4\times 3ac}$$
$$=5\sqrt{64a^2b^4}\times\sqrt{3ac}$$
$$=5\times\pm 8ab^2\sqrt{3ac}$$
$$=\pm 40ab^2\sqrt{3ac}$$

$$4\sqrt{1024a^7b^6c^4}=4\sqrt{1024a^6b^4c^4\times ab}$$
$$=4\times\pm 32a^3b^2c^2\sqrt{ab}$$
$$=\pm 128a^3b^2c^2\sqrt{ab}.$$

(94.) The quantity which stands without the radical sign is called the *coefficient* of the radical; and when there is no coefficient, the *unit* is always understood.

From the preceding, to simplify a radical of the second degree, we derive the following

Rule.

Decompose the quantity under the radical sign into two factors, one of which shall be the greatest perfect square. Extract the square root of this perfect square, multiply the root by the coefficient of the radical, and under the radical sign write the other factor.

Examples.

1. Reduce $3a\sqrt{4x^3-4a^2x^2}$ to its simplest form.

$$3a\sqrt{4x^3-4a^2x^2}=3a\sqrt{4x^2(x-a^2)}$$
$$=\pm 6ax\sqrt{x-a^2},\ \textit{Ans.}$$

2. Reduce $2\sqrt{150a^3b^2c}$ to its simplest form.
Ans. 10ab√6ac

3. Reduce $7\sqrt{567a^5b^6c^4}$ to its simplest form.
Ans.

4. Reduce $\sqrt{300a^4b^3c^5d}$ to its simplest form.
Ans.

5. Reduce $4\sqrt{605x^2y^3z^4}$ to its simplest form.
Ans.

6. Reduce $6\sqrt{432a^2x^3y^4}$ to its simplest form.
Ans.

7. Reduce $2a\sqrt{945mx^2z^3}$ to its simplest form.
Ans.

8. Reduce $\sqrt{6a^2+a^2b}$ to its simplest form.
Ans. $a\sqrt{6+b}$.

9. Reduce $7\sqrt{243a^3b^3c}$ to its simplest form.
Ans.

10. Reduce $8\sqrt{108a^3b^3}$ to its simplest form.

Ans.

When the terms under the radical sign are *fractional*, it is better to make the denominator a perfect square.

11. Reduce $\sqrt{\dfrac{50}{147}}$ to its simplest form.

$$\sqrt{\frac{50}{147}}=\sqrt{\frac{150}{441}}=\sqrt{\frac{25}{441}\times 6}=\frac{5}{21}\sqrt{6}.$$

12. Reduce $\sqrt{\dfrac{2}{3}}$ to its simplest form.

Ans. $\dfrac{1}{3}\sqrt{6}$.

13. Reduce $\sqrt{\dfrac{27}{50}}$ to its simplest form.

Ans. $\dfrac{3}{10}\sqrt{6}$.

14. Reduce $\sqrt{\dfrac{3}{5}}$ to its simplest form.

Ans. $\dfrac{1}{5}\sqrt{15}$.

15. Reduce $\sqrt{\dfrac{5}{16}}$ to its simplest form.

Ans.

16. Reduce $\sqrt{\dfrac{7}{8}}$ to its simplest form.

Ans. $\dfrac{1}{4}\sqrt{14}$.

17. Reduce $\sqrt{\dfrac{5}{27}}$ to its simplest form.

Ans. $\dfrac{1}{9}\sqrt{15}$.

18. Reduce $\sqrt{\frac{54}{147}}$ to its simplest form.

Ans. $\frac{3}{7}\sqrt{2}$.

19. Reduce $\sqrt{\frac{2a^2}{3}}$ to a simpler form.

Ans. $\frac{a}{3}\sqrt{6}$.

20. Reduce $\sqrt{\frac{147}{845}}$ to a simpler form.

Ans. $\frac{7}{65}\sqrt{15}$.

(95.) *Two radicals of the second degree are similar when the quantities under the radical sign are the same in both.* Thus, $\frac{1}{3}\sqrt{6}$ and $\frac{3}{10}\sqrt{6}$ are similar radicals. So are $5\sqrt{7}$ and $6\sqrt{7}$.

If we put the $\sqrt{7}=x$, the above will become $5x$ and $6x$, and their sum $=11x$; or, by replacing the value of x, we have $11\sqrt{7}$, and their difference $=\sqrt{7}$.

Hence, to add or subtract similar radicals, we have the following

RULE.

For addition,

Add the coefficients, and to their sum annex the common radical.

For subtraction,

Subtract the coefficient of the subtrahend from the coefficient of the minuend, and to the remainder annex the common radical.

When the radicals are dissimilar, reduce them to imilar radicals (if possible), and proceed as above. If they can not be so reduced, they must be connected by the proper signs.

EXAMPLES.

1. Required the sum of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{147}{845}}$.

Here $\sqrt{\frac{3}{5}}=\sqrt{\frac{15}{25}}=\frac{1}{5}\sqrt{15}$,

and $\sqrt{\frac{147}{845}}=\sqrt{\frac{49}{169}\times\frac{3}{5}}=\frac{7}{13}\sqrt{\frac{3}{5}}=\frac{7}{13}\sqrt{\frac{15}{25}}=\frac{7}{13}\times\frac{1}{5}\sqrt{15}=\frac{7}{65}\sqrt{15}$;

then $\frac{1}{5}\sqrt{15}+\frac{7}{65}\sqrt{15}=\left(\frac{1}{5}+\frac{7}{65}\right)\sqrt{15}=\frac{4}{13}\sqrt{15}$, *Ans.*

2. Required the sum of $\sqrt{27}$ and $\sqrt{48}$.

Ans. $7\sqrt{3}$.

3. Add together $\sqrt{72}$ and $\sqrt{128}$.

Ans. $14\sqrt{2}$.

4. Required the sum and difference of

$\sqrt{405}$ and $\sqrt{180}$.

Ans. Sum$=15\sqrt{5}$, diff. $=3\sqrt{5}$.

(96.) *The product of two radicals of the second degree is equal to the square root of the product of the quantities under the radical signs. When there are coefficients, place their product for the coefficient.*

The quotient arising from dividing one radical by another is equal to the square root of the quotient of the quantities under the radical signs.

When the radicals have coefficients, their quotient is the coefficient.

EXAMPLES.

1. Multiply $\frac{3}{4}\sqrt{\frac{7}{8}}$ by $\frac{5}{6}\sqrt{\frac{8}{9}}$.

Here $\frac{3}{4}\times\frac{5}{6}=\frac{5}{8}=$coefficient,

and $\sqrt{\frac{7}{8}}\times\sqrt{\frac{8}{9}}=\sqrt{\frac{7}{9}}$;

$$\therefore\ \frac{5}{8}\sqrt{\frac{7}{9}}=\text{the product.}$$

But by simplifying, we have

$$\frac{5}{8}\sqrt{\frac{7}{9}}=\frac{5}{8}\sqrt{\frac{1}{9}\times 7}=\frac{5}{8}\times\frac{1}{3}\sqrt{7}=\frac{5}{24}\sqrt{7},\ \textit{Ans.}$$

2. Divide $\frac{3}{4}\sqrt{\frac{7}{8}}$ by $\frac{5}{6}\sqrt{\frac{8}{9}}$.

Here $\frac{3}{4}\div\frac{5}{6}=\frac{9}{10}=$coefficient,

and $\sqrt{\frac{7}{8}}\div\sqrt{\frac{8}{9}}=\sqrt{\frac{7}{8}\div\frac{8}{9}}=\sqrt{\frac{63}{64}}=\sqrt{\frac{9}{64}\times 7}=\frac{3}{8}\sqrt{7}$;

$$\therefore\ \frac{9}{10}\times\frac{3}{8}\sqrt{7}=\frac{27}{80}\sqrt{7},\ \textit{Ans.}$$

3. Multiply $\sqrt{\frac{3}{5}}$ by $\sqrt{\frac{5}{27}}$. *Ans.* $\frac{1}{3}$.

4. Multiply $5\sqrt{8}$ by $3\sqrt{5}$.

Ans. $30\sqrt{10}$.

5. Divide $8\sqrt{18}$ by $2\sqrt{3}$.

Ans. $4\sqrt{6}$.

6. Divide $21\sqrt{10}$ by $3\sqrt{5}$. *Ans.* $7\sqrt{2}$.

(97.) We have seen how a radical of the second degree is reduced to a simpler form. We shall now show in what manner the coefficient of a radical can be put under the radical sign.

Take the expression $5x\sqrt{3y}$,

we may write $5x=\sqrt{25x^2}$;

then $5x\sqrt{3y}=\sqrt{25x^2}\times\sqrt{3y}=\sqrt{25x^2\times 3y}=\sqrt{75x^2y}$.

Hence, *to pass the coefficient of a radical of the second degree under the radical sign, we must square it, and multiply it by the quantity under the radical sign.*

The object of this transformation is to obtain an approximate value of the radical.

Another transformation is extensively used; that is, to render the denominator of a fraction of the form

$$\frac{a}{\sqrt{b}+\sqrt{c}} \text{ or } \frac{a}{\sqrt{b}-\sqrt{c}},$$

in which b and c are imperfect squares, rational.

If we multiply both members of the first by $\sqrt{b}-\sqrt{c}$, or the second by $\sqrt{b}+\sqrt{c}$, we shall effect the purpose; for the product of the sum and difference of two quantities is equal to the difference of their squares.

Therefore $\dfrac{a\sqrt{b}-a\sqrt{c}}{b-c}$ and $\dfrac{a\sqrt{b}+a\sqrt{c}}{b-c}$

are the quantities with rational denominators.

EXAMPLES.

1. Reduce $\dfrac{3\sqrt{15}-4\sqrt{5}}{\sqrt{15}+\sqrt{5}}$ to a fraction having a rational denominator.

Here $\frac{3\sqrt{15}-4\sqrt{5}}{\sqrt{15}+\sqrt{5}}=\frac{(3\sqrt{15}-4\sqrt{5})(\sqrt{15}-\sqrt{5})}{(\sqrt{15}+\sqrt{5})(\sqrt{15}-\sqrt{5})}$

$$=\frac{65-7\sqrt{75}}{15-5}$$

$$=\frac{13-7\sqrt{3}}{2}.$$

For $7\sqrt{75}=7\sqrt{25\times3}=35\sqrt{3}.$

2. Reduce $\frac{3}{\sqrt{5}-\sqrt{x}}$ to a fraction having a rational denominator.

Ans. $\frac{3\sqrt{5}+3\sqrt{x}}{5-x}.$

3. Reduce $\frac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction having a rational denominator.

Ans. $\frac{3\sqrt{2}+2}{7}.$

The object of the preceding transformation is to obtain the *numerical value* of the fraction to a greater degree of exactness.

4. Find the value of $\frac{3+2\sqrt{7}}{5\sqrt{12}-6\sqrt{5}}$ to within .001.

Here $\frac{3+2\sqrt{7}}{5\sqrt{12}-6\sqrt{5}}=\frac{(3+2\sqrt{7})(5\sqrt{12}+6\sqrt{5})}{(5\sqrt{12}-6\sqrt{5})(5\sqrt{12}+6\sqrt{5})}$

$$=\frac{15\sqrt{12}+18\sqrt{5}+10\sqrt{84}+12\sqrt{35}}{300-180}$$

$$=\frac{\sqrt{2700}+\sqrt{1620}+\sqrt{8400}+\sqrt{5040}}{120}$$

$$=\frac{51.96+40.24+91.60+70.99}{120}$$

$$=2.123$$

5. Find the value of $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$. *Ans.* 0.5595.

6. Find the value of $\frac{9+2\sqrt{10}}{9-2\sqrt{10}}$. *Ans.* 5.7278.

7. Find the value of $\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}$. *Ans.* 3.1.

8. Find the value of $\sqrt{\frac{3}{5}}$, or its equal $\frac{\sqrt{3}}{\sqrt{5}}$.

Ans. .77.

(98.) *To raise a radical to any power.*

Let us take the $x^{\frac{1}{m}}$ and raise it to the nth power, we have $x^{\frac{1}{m}}\times x^{\frac{1}{m}}\times x^{\frac{1}{m}}$ . . . . to n factors $=x^{\frac{n}{m}}, =\sqrt[m]{x^n}$.

Hence we have the following

RULE.

If the radical has a coefficient, raise it to the given power; then raise the quantity under the radical sign to the given power, and affect this quantity with that radical sign having the original index.

1. What is the cube of $\frac{1}{2}\sqrt{2ax}$?

Here $(\frac{1}{2})^3=\frac{1}{8}$ and $(\sqrt{2ax})^3=(2ax)^{\frac{3}{2}}=2ax\times(2ax)^{\frac{1}{2}}$;

$\therefore\ (\frac{1}{2}\sqrt{2ax})^3=\frac{1}{4}ax\sqrt{2ax}$, *Ans.*

2. Find the square of $\sqrt{a}-\sqrt{b}$.

$$\begin{array}{l} \sqrt{a}-\sqrt{b} \\ \underline{\sqrt{a}-\sqrt{b}} \\ a-\sqrt{ab} \\ \underline{\quad -\sqrt{ab}+b} \\ a-2\sqrt{ab}+b, \textit{ Ans.} \end{array}$$

3. Find the square of $3\sqrt[3]{3}$.

Ans. $9\sqrt[3]{9}$.

4. Find the fourth power of $-\sqrt[3]{a^2}$.

Ans. $a^2\sqrt[3]{a^2}$.

5. Find the cube of $a-\sqrt{b}$.

Ans. $a^3-3a^2\sqrt{b}+3ab-b\sqrt{b}$.

6. Required the square of $3+\sqrt{5}$.

Ans. $14+6\sqrt{5}$.

7. Required the square of $\sqrt[4]{b^4+x^4}$.

Ans. $\sqrt{b^4+x^4}$.

8. Required the square of $p-\sqrt{-q}$.

Ans. $p^2-2p\sqrt{-q}-q$.

(99.) *To reduce radicals to a common index.*

Let us take $\sqrt{a}$ and $\sqrt[3]{b}$, and reduce them to a common index.

Here $\sqrt{a}=a^{\frac{1}{2}}$ and $\sqrt[3]{b}=b^{\frac{1}{3}}$. Now the fractions $\frac{1}{2}$ and $\frac{1}{3}$, reduced to the least common denominator, are $\frac{3}{6}$ and $\frac{2}{6}$; therefore

$$\sqrt{a}=a^{\frac{3}{6}} \text{ and } \sqrt[3]{b}=b^{\frac{2}{6}};$$

$\therefore$ $\sqrt[6]{a^3}$ and $\sqrt[6]{b^2}$ are the radicals required.

Hence we have the following

Rule.

Reduce the indices of the given quantities to fractions having the least common denominator; raise the quantities under the radical sign respectively to the power denoted by the numerator of its index, and place the common radical over each of them.

1. Reduce $a^{\frac{1}{4}}$ and $x^{\frac{1}{3}}$ to the same radical sign.

Ans. $\sqrt[12]{a^3}$ and $\sqrt[12]{x^4}$.

2. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to radicals having the same index.

Ans. $\sqrt[mn]{a^m}$ and $\sqrt[mn]{b^n}$.

3. Reduce $\sqrt[3]{a}$ and $\sqrt[5]{b}$ to radicals having the same index.

Ans. $\sqrt[15]{a^5}$ and $\sqrt[15]{b^3}$.

4. Reduce $3\sqrt[3]{2}$ and $2\sqrt{5}$ to radicals of the same index.

Ans. $3\sqrt[6]{4}$ and $2\sqrt[6]{125}$.

5. Reduce $\sqrt[3]{xy}$ and $\sqrt[4]{ax}$ to radicals having the same index.

Ans. $\sqrt[12]{x^4y^4}$ and $\sqrt[12]{a^3x^3}$.

(100.) Extraction of the Square Root of Polynomials.

If we consider the polynomial whose root is to be extracted, as arranged according to the highest or lowest power of a certain letter; then, if the polynomial is a perfect square, its first term will be a perfect square, and the square root of the first term will be the first term of the root.

Let us suppose P to be the polynomial thus arranged, and that r', r'', r''', &c., are the successive terms of the root; then

$$\begin{aligned}P&=(r'+r''+r'''+, \&c.)^2\\&=r'^2+2r'r''+r''^2+2r'r'''+2r''r'''+r'''^2+, \&c.\end{aligned}$$

By transposing r'^2 to the first member, and putting $P-r'^2=R'$, we have

$$P-r'^2=R'=2r'r''+r''^2+2r'r'''+2r''r'''+r'''^2+, \&c.$$

If we now divide the first term of the second member by $2r'$, we obtain r''.

Again, transpose $2r'r''+r''^2$, or its equal $(2r'+r'')r''$, to the first member,

$$R'-(2r'+r'')r''=2r'r'''+2r''r'''+r'''^2+, \&c.$$

We may put the first member equal to R''; we shall then have

$$R''=2r'r'''+2r''r'''+r'''^2+, \&c.$$

Dividing by $2r'$, we obtain r'''; and, by transposing, $2r'r'''+2r''r'''+r'''^2$, or its equal $(2r'+2r''+r''')$, we have

$$R''-(2r'+2r''+r''')r'''=, \&c.,$$

from which we derive the following

Rule.

Arrange the polynomial with reference to a certain letter, then extract the square root of the first term on the left of the polynomial for the first term of the root; subtract its square from the given polynomial.

Divide the first term of the remainder by twice the first term of the root for the second term of the root.

Multiply twice the first term plus the second by the second, and subtract the product from the first remainder.

Divide the first term of this remainder by twice the first term of the root for the third term of the root.

Multiply twice the first term plus twice the second, plus the third by the third, and subtract the product from the second remainder. From which another term of the root may be obtained.

When the first term of any one of the remainders is not exactly divisible by twice the first term of the root, the polynomial is not a perfect square.

Examples.

1. Extract the square root of $4x^4+6x^3+22\frac{1}{4}x^2+15x+25$.

$$\begin{array}{rr|l}
 & 4x^4+6x^3+22\frac{1}{4}x^2+15x+25 & 2x^2+\ \frac{3}{2}x+5 \\
 & 4x^4 \qquad\qquad\qquad\qquad\quad & \overline{4x^2+\ \frac{3}{2}x} \\
\cline{2-2}
R' = & 6x^3+22\frac{1}{4}x^2+15x+25 & \frac{3}{2}x \\
 & 6x^3+\ 2\frac{1}{4}x^2 \qquad\qquad\quad & \overline{6x^3+2\frac{1}{4}x^2} \\
\cline{2-2}
R'' = & 20x^2\ +15x+25 & \overline{4x^2+\ 3x+5} \\
 & 20x^2\ +15x+25 & 5 \\
\cline{2-3}
R''' = & 0. \qquad\quad & 20x^2+15x+25
\end{array}$$

2. Extract the square root of $x^4+4ax^3+6a^2x^2+4ax^3+a^4$.

Ans. $x^2+2ax+a^2$.

3. Extract the square root of $a^4-2a^3+\frac{3}{2}a^2-\frac{1}{2}a+\frac{1}{16}$.

Ans. $a^2-a+\frac{1}{4}$.

4. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

Ans. $2a^2+3ax+x^2$.

5. Extract the square root of $9x^4+12x^3+34x^2+20x+25$.

Ans. $3x^2+2x+5$.

6. Extract the square root of $a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2$.

Ans. $a+b+c+d$.

7. Extract the square root of $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$.

Ans. $a^3-3a^2x+3ax^2-x^3$.

8. Extract the square root of $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$. *Ans.* $a^2+6ab+9b^2$.

9. Extract the square root of $a^2+2ab+b^2+2ac+2bc+c^2$. *Ans.* $a+b+c$.

10. Extract the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$. *Ans.* a^2-ax+x^2.

11. Extract the square root of $x^2+2xy+y^2+6xz+6yz+9z^2$. *Ans.* $x+y+3z$.

12. Extract the square root of $x^6+4x^5+2x^4+9x^2-4x+4$. *Ans.* x^3+2x^2-x+2.

13. Extract the square root of $16a^4+24a^3+89a^2+60a+100$. *Ans.* $4a^2+3a+10$.

14. Extract the square root of $x^4+4x^2y+4y^2-4x^2-8y+4$. *Ans.* x^2+2y-2.

15. Extract the square root of $1-4x+4x^2+2y-4xy+y^2$. *Ans.* $1-2x+y$.

16. Extract the square root of $4a^4-4a^3+13a^2-6a+9$. *Ans.* $2a^2-a+3$.

17. Extract the square root of $4a^4-16a^3+24a^2-16a+4$. *Ans.* $2a^2-4a+2$.

(101.) *A binomial can never be a perfect square.* For the square of a monomial is a monomial, and the square of the least polynomial is a trinomial. But

A trinomial is a perfect square when, being arranged, its first and third terms are perfect squares, and its second term is equal to twice the product of their square roots.

Hence, to extract the square root of a trinomial which is a perfect square,

Extract the roots of the two extreme terms, and unite them by the sign of the second term.

When a polynomial is not a perfect square, the expression for its square root may frequently be simplified by the rule for simplifying a radical of the second degree.

Thus, take the expression

$$\sqrt{x^3y-x^2y+\frac{xy}{4}}.$$

We can decompose this into two factors

$$\sqrt{x^2-x+\tfrac{1}{4}}\times\sqrt{xy},$$

and by extracting the root of the perfect square, we shall have

$$(x-\tfrac{1}{2})\sqrt{xy}.$$

(102.) *To extract the* n*th root of a polynomial.*

Rule.

Arrange the polynomial according to the powers of one of its letters, so that the highest power of that letter shall be the first term; extract the n*th root of this first term for the first term of the root.*

Subtract the n*th power of this first term of the root from the given polynomial, and divide the first term of the remainder by the first term of the root raised to the* n−1 *power, multiplied by* n, *for the second term of the root.*

Subtract the n*th power of the algebraic sum of the terms found from the given polynomial, and, using the same divisor, continue the process until all the successive terms of the root are found.*

Examples.

1. What is the cube root of

$$\begin{array}{rl} & x^6+6x^5-40x^3+96x-64 \,|\, x^2+2x-4 \\ & x^6 \\ \text{Divisor, } 3x^4 & 6x^5-40x^3+96x-64 = \text{1st rem.} \\ & x^6-6x^5+12x^4+8x^3 \quad =(x^2+2x)^3 \\ 3x^4 & -12x^4+88x^3-64 = \text{2d rem.} \\ & x^6+6x^5-40x^3+96x-64 \\ & 0\,? \end{array}$$

2. What is the fourth root of

$$\begin{array}{rl} & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4 \,|\, 2a-3x \\ & 16a^4 \\ 4\times(2a)^3=32a^3 & -96a^3x \\ & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4\,? \end{array}$$

3. What is the cube root of

$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1\,?$$

Ans. x^2-2x+1.

4. What is the fifth root of

$$32x^5-80x^4+80x^3-40x^2+10x-1\,?$$

Ans. $2x-1$.

To apply this general rule to the extraction of the cube root of numbers, we must point off the given power into periods of three figures each, beginning at the unit. Thus, extract the cube root of 12167.

$$\begin{array}{rl} & 12\cdot167(23 \\ a=2 \;\therefore\; a^3= & 8 \\ \text{Divisor,} \quad 3a^2=12 & |41 \\ & 12167=(23)^3. \end{array}$$

CHAPTER V.

EQUATIONS OF THE SECOND DEGREE.

(103.) If an equation, when cleared of fractional exponents, contain only the *first power* of the unknown quantity, it is called an equation of the *first degree.*

If the unknown quantity rises to the *second power,* it is called an equation of the *second degree.*

If the equation contains two unknown quantities, it is of the second degree when the greatest sum of the exponents with which the unknown quantities are affected in any term is equal to 2.

(104.) Equations of the second degree are divided into two classes, *complete* and *incomplete.*

A *complete equation* involves the first and second powers of the unknown quantity and known terms.

An *incomplete equation* involves the second power of the unknown quantity only and known terms.

(105.) An incomplete equation can always be reduced to the form

$$x^2=a,$$

and has, from this fact, sometimes been called an equation of *two terms,* or a *binomial equation.*

Thus, $$\frac{x^2}{5}+3x^2=5x^2-\frac{x^2}{10}-170,$$

when cleared of fractions, and transposed, becomes

$$17x^2=1700,$$

or $$x^2=100.$$

G 2

We find no difficulty in solving equations of this form, for

$$x = \pm \sqrt{a}.$$

If a is a perfect square, as in the example above, then

$$x = \pm 10.$$

If a is not a perfect square, it must be reduced to its simplest form (*Art.* 94).

That in the equation

$$x^2 = a,$$

x may be equal to $+a$ or $-a$, is evident, for if we transpose a to the first member, we shall have

$$x^2 - a = 0.$$

But the first member may be put, by factoring it, under the form

$$(x + \sqrt{a})(x - \sqrt{a}) = 0.$$

Now, when the product of two factors is equal to zero, the factors themselves may be put each equal to zero, so we shall have

$$x + \sqrt{a} = 0,$$

or $$x = -\sqrt{a},$$

and $$x - \sqrt{a} = 0;$$

$$\therefore\ x = +\sqrt{a}.$$

Again, if we have the equation

$$x^2 = -n,$$

where n is a *positive* quantity, then

$$\begin{aligned} x &= \pm\sqrt{-n} \\ &= \pm\sqrt{n} \times \sqrt{-1}. \end{aligned}$$

Therefore the two roots of an *incomplete* or *binomial* equation are either both *real* or both *imaginary.*

(106.) If the value of the unknown quantity be sub-

stituted in the equation, it will verify the equation; that is, it will make the first member equal to the second.

In an incomplete equation of the second degree there are two values of the unknown quantity numerically equal to each other, but with contrary signs.

1. Let us take, as an example,

$\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$ to find the value of x.

By transposition, we have

$$\sqrt[3]{(a+x)}=b-\sqrt[3]{(a-x)}.$$

If we cube both members of this equation, we shall have

$$a+x=b^3-3b^2\sqrt[3]{(a-x)}+3b\sqrt[3]{(a-x)^2}-(a-x).$$

Transpose and reduce, we obtain

$$3b^2\sqrt[3]{(a-x)}=b^3+3b\sqrt[3]{(a-x)^2}-2a.$$

Divide by b,

$$3b\sqrt[3]{(a-x)}=\frac{b^3-2a}{b}+3\sqrt[3]{(a-x)^2}.$$

If, in the first member of this equation, we substitute for b its value in the original equation, we shall have

$$3\sqrt[3]{(a^2-x^2)}+3\sqrt[3]{(a-x)^2}=\frac{b^3-2a}{b}+3\sqrt[3]{(a-x)^2}.$$

The second terms in each member cancel each other, and

$$3\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{b},$$

or $$\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{3b}.$$

Now, by cubing both members, we have

$$a^2-x^2=\left(\frac{b^2-2a}{3b}\right)^2,$$

or $$x^2=a^2-\left(\frac{b^2-2a}{3b}\right)^2;$$

$$\therefore\ x=\pm\sqrt{a^2-\left(\frac{b^2-2a}{3b}\right)^2}.$$

2. Take, as a second example,

$$\sqrt{a^2-x^2}+x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}$$

to find the value of x.

By transposing the second term, we have

$$\sqrt{(a^2-x^2)}=a^2\sqrt{(1-x^2)}-x\sqrt{a^2-1}.$$

Squaring both members, gives

$$a^2-x^2=a^4-a^4x^2-2a^2x\sqrt{(1-x^2)(a^2-1)}+a^2x^2-x^2.$$

By transposition and reduction,

$$2x\sqrt{(1-x^2)(a^2-1)}=a^2-a^2x^2-1+x^2.$$

The second member, factored, becomes

$$(a^2-1)(1-x^2);$$

$$\therefore\ 2x\sqrt{(a^2-1)(1-x^2)}=(a^2-1)(1-x^2).$$

Squaring both members of this equation, we obtain

$$4x^2(a^2-1)(1-x^2)=(a^2-1)^2(1-x^2)^2.$$

Divide both members by $(a^2-1)(1-x^2)$, and we have

$$4x^2=(a^2-1)(1-x^2),$$

or $$4x^2=a^2-a^2x+x^2-1;$$

then $$3x^2+a^2x^2=a^2-1,$$

or $$(3+a^2)x^2=a^2-1$$

$$x^2=\frac{a^2-1}{3+a^2};$$

$$\therefore\ x=\pm\sqrt{\frac{a^2-1}{3+a^2}}.$$

3. Take the example $a^2+x^2=\sqrt{b^4+x^4}$ to find the value of x.

Square both members, and we have

$$a^4+2a^2x^2+x^4=b^4+x^4.$$

Transpose and reduce,

$$2a^2x^2=b^4-a^4$$

$$x^2=\frac{b^4-a^4}{2a^2};$$

$$\therefore\ x=\pm\sqrt{\frac{b^4-a^4}{2a^2}}.$$

(107.) Hence, for the solution of an incomplete equation, we have the following

RULE.

If the equation contains but one radical quantity, transpose all the other terms to one member; then involve both members to the power indicated by the index of the radical.

If there is more than one radical sign over the quantity, the operation must be repeated; and if there are several radical quantities, let the most complex form one member of the equation by itself, and then proceed as above.

EXAMPLES.

1. Given $\frac{9}{2+2x}+\frac{9}{2-2x}=25$ to find the value of x.

Ans. $x=\pm\frac{4}{5}$.

2. Given $\sqrt{\frac{x+1}{x-1}}+\sqrt{\frac{x-1}{x+1}}=a$ to find the value of x.

Ans. $x=\pm\frac{a}{\sqrt{a^2-4}}$.

3. Given $\sqrt{a+x}+\sqrt{a-x}=b$ to find the value of x.

Ans. $x=\pm\frac{b}{2}\sqrt{4a-b^2}$.

4. Given $\sqrt{a^2+ax}=a-\sqrt{a^2-ax}$ to find the value of x.

Ans. $x=\pm\frac{a}{2}\sqrt{3}$.

5. Given $\frac{\sqrt{9x}-4}{\sqrt{x}+2}=\frac{15+\sqrt{9x}}{\sqrt{x}+40}$ to find the value of x.

Ans. $x=4$.

6. Given $\sqrt{x}+\sqrt{x-9}=\frac{36}{\sqrt{x-9}}$ to find the value of x.

Ans. $x=25$.

7. Given $\sqrt{a}+\sqrt{x}=\sqrt{ax}$ to find the value of x.

Divide each term of the equation by $\sqrt{x}$, we have

$$\frac{\sqrt{a}}{\sqrt{x}}+1=\sqrt{a},$$

or
$$\frac{\sqrt{a}}{\sqrt{x}}=\sqrt{a}-1$$

$$\sqrt{a}=\sqrt{x}(\sqrt{a}-1),$$

or
$$\sqrt{x}=\frac{\sqrt{a}}{\sqrt{a}-1}.$$

$$\therefore\ x=\frac{a}{(\sqrt{a}-1)^2}.$$

(108.) If the equation can be reduced to the form $\sqrt[n]{x}=a$, then the value of x can be found by raising both members to the nth power, as in example 7 above.

(109.) When the equation can be reduced to the form

$$x^{\frac{m}{n}}=a.$$

First involve both members to the n*th power, and then extract the* m*th root; or first extract the* m*th root, then involve to the* n*th power.*

8. Given $\left.\begin{array}{l} x^{\frac{2}{3}}+y^{\frac{2}{3}}=13 \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=\ 5 \end{array}\right\}$ to find the values of x and y.

Squaring the second equation, we have

$$x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=25,$$

but $$x^{\frac{2}{3}} \qquad +x^{\frac{2}{3}}=13.$$

Subtract, $$2x^{\frac{1}{3}}y^{\frac{1}{3}}=12.$$

Subtract this equation from the first equation,

$$x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=1,$$

and extract the square root of both members,

$$x^{\frac{1}{3}}-y^{\frac{1}{3}}=\pm 1,$$

but $$x^{\frac{1}{3}}+y^{\frac{1}{3}}=\ 5;$$

$\therefore$ by addition, $$2x^{\frac{1}{3}}=\ 6 \text{ or } 4,$$

or $$x^{\frac{1}{3}}=\ 3 \text{ or } 2;$$

$$\therefore\ x=27 \text{ or } 8.$$

If we subtract instead of adding, we shall have

$$2y^{\frac{1}{3}}=4 \text{ or } 6,$$

or $$y^{\frac{1}{3}}=2 \text{ or } 3;$$

$$\therefore\ y=8 \text{ or } 27.$$

If the value of x be taken 27, then $y=8$, or if $x=8$, then $y=27$.

9. Given $\left.\begin{array}{l} x^3-xy^2-x^2y+y^3=\ 3xy \\ x^6-x^2y^4-x^4y^2+y^6=45x^2y^2 \end{array}\right\}$ to find x and y.

Dividing the second equation by the first gives

$$x^3+x^2y+xy^2+y^3=15xy,$$

but (1), $$x^3-x^2y-xy^2+y^3=\ 3xy.$$

Adding, $$2x^3+\ 2y^3=18xy,$$

or $$x^3+\ \ y^3=\ 9xy.$$

Subtracting, $$2x^2y+2xy^2=12xy,$$

or $$x^2y+\ xy^2=\ 6xy,$$

or $$xy(x+y)=\ 6xy,$$

or $$x+y=\ 6.$$

Cubing both members of this last equation, have

$$x^3+3x^2y+3xy^2+y^3=216;$$

but $$x^3 \qquad\qquad +y^3=9xy.$$

Subtract, $$3x^2y+3xy^2=216-9xy.$$

Factoring this first member gives

$$3(x+y)xy=216-9xy.$$

But $$x+y=6;$$

$$\therefore\ 18xy=216-9xy,$$

or $$27xy=216;$$

$$\therefore\ xy=8.$$

Again, $$x^2+2xy+y^2=36,$$

and $$4xy \qquad =32.$$

Subtract, $$x^2-2xy+y^2=4.$$

Extracting the square root of both members, we tain

$$x-y=\pm 2;$$

but $$x+y=6.$$

By addition, $$2x=8 \text{ or } 4;$$

$$\therefore\ x=4 \text{ or } 2.$$

And by subtraction, $$2y=4 \text{ or } 8;$$

$$\therefore\ y=2 \text{ or } 4.$$

10. Given $\left.\begin{array}{r}3x+3y=5x\\ xy=6\end{array}\right\}$ to find the value of x and y.

Ans. $x=\pm3$, $y=\pm2$.

11. Given $\left.\begin{array}{r}x^2+y^2=\dfrac{13}{x-y}\\ xy=\dfrac{6}{x-y}\end{array}\right\}$ to find x and y.

Ans. $x=3$ or -2,
$y=2$ or -3.

12. Given $\left.\begin{array}{r}x^2+xy=\ \ 6\\ y^2+xy=-5\end{array}\right\}$ to find x and y.

Ans. $x=\pm6$, $y=\mp5$.

Problems.

1. There are two numbers which are in the proportion of 3 to 2, the difference of whose fourth powers is to the sum of their cubes as 26 to 7. What are the numbers?

Let $3x$ and $2x$ represent the two numbers.

Then $$65x^4 : 35x^3 :: 26 : 7,$$
or $$13x : 7 :: 26 : 7;$$
$$\therefore\ 13x=26,$$
and $$x=2;$$
hence the numbers are 6 and 4.

2. The sum of the cubes of two numbers is 35, and the sum of their ninth powers is 20195 What are the numbers?

Let x and y represent the numbers.

Then $$x^3+y^3=35,$$
and $$x^9+y^9=20195.$$

Dividing the second equation by the first, member by member, we have

$$x^6-x^3y^3+x^6=577,$$

and, squaring the first equation,

$$x^6+2x^3y^3+y^6=1225.$$

Subtracting, $$3x^3y^3=648,$$

or $$x^3y^3=216;$$

$$\therefore\ xy=6.$$

If we now subtract x^3y^3 from the first member of equation (3), and its value, 216, from the second member of the same, we shall obtain

$$x^6-2x^3y^3+y^6=361,$$

and, extracting the root of both members, gives

$$x^3-y^3=19,$$

but $$x^3+y^3=35;$$

$$\therefore\ 2x^3=54$$

$$x^3=27,$$

and $$x=3.$$

If we subtract, we shall have

$$2y^3=16$$

$$y^3=8$$

$$y=2.$$

3. Find two numbers such that the product of the greater and square of the less may be equal to 36, and the product of the less and square of the greater may be 48.

Ans. 4 and 3.

4. What number is that whose half, multiplied by its third part, gives 864?

Ans. 72.

5. What two numbers are as m to n, the difference of whose squares is a?

Ans. $\dfrac{m\sqrt{a}}{\sqrt{m^2-n^2}}$ and $\dfrac{n\sqrt{a}}{\sqrt{m^2-n^2}}$.

6. A certain capital is put out at 4 per cent.; if we multiply the number of dollars in the capital by the number of dollars in the interest for 5 months, we obtain $117041\frac{2}{3}$. What is the capital?

Ans. \$2650.

7. Find three numbers such that the product of the first and second is 6, that of the first and third is 10, and the sum of the squares of the second and third is 34. *Ans.* 2, 3, 5.

8. What two numbers are they the sum of whose squares is a, and the difference of whose squares is b?

Ans. $\sqrt{\frac{1}{2}(a+b)}$,
$\sqrt{\frac{1}{2}(a-b)}$.

9. There are two numbers whose product is 300, and the difference of their cubes is thirty-seven times the cube of their difference. What are the numbers?

Ans. 20 and 15.

10. Find two numbers in the proportion of m to n, the sum of whose squares is a.

Ans. $\dfrac{m\sqrt{a}}{\sqrt{m^2+n}}$ and $\dfrac{n\sqrt{a}}{\sqrt{m^2+n^2}}$.

11. What two numbers are as 8 to 5, the sum of whose squares is 801?

Ans. 24 and 15.

12. A and B carried one 100 eggs between them to market, and each received the same sum. If A had carried as many as B, he would have received 18 pence for them; and if B had only taken as many as A, he would have received 8 pence. How many had each?

Ans. A 40, and B 60.

13. There are two squares whose areas are as 25 to

9, and a side of the greater is 10 yards longer than a side of the less. What are the lengths of the sides?

Ans. Side of greater, $=25$,
" less, $=15$.

14. The sum of two numbers is 8, and the sum of their cubes is 152. What are the numbers?

Ans. 5 and 3.

Put $2s=8=$sum, and $2d=$difference, then $s+d=$ greater, and $s-d=$less.

15. From two places at an unknown distance from each other, two travelers, A and B, set out to meet. On coming together, it appeared that A had traveled 18 miles more than B, and that A could have gone B's journey in $15\frac{3}{4}$ hours, but B would have been 28 hours in performing A's journey. What was the distance between the two places?

After dividing each member of the equation by 7, *they are both perfect squares.*

(110.) OF COMPLETE EQUATIONS OF THE SECOND DEGREE.

An equation of the second degree is said to be complete when it is of the form

$$x^2+2ax=b;$$

that is, when it can be reduced to three terms, the first containing the square of the unknown quantity, the second the unknown quantity to the first power, the third the known quantity or absolute term.

The method of reduction is the same as that given for equations of the first degree, the only difficulty being to obtain the value of x from the equation

$$x^2+2ax=b.$$

The first member being a binomial, is not a perfect square. We must, therefore, consider in what way we can render it so. We may consider x^2 as the square of the first term of the root, and in this case $2ax$ must represent twice the product of x, the first term of the root by the second term, hence this second term must be a; and, in fact, the square of $x+a$ is found to be

$$x^2+2ax+a^2.$$

Now $x^2+2ax+a^2$ being a perfect square, whose root is $x+a$, if we resume our equation

$$x^2+2ax=b,$$

we have only to add a^2 to both members, which gives

$$x^2+2ax+a^2=b+a^2.$$

The first member being a perfect square, and the second containing only known quantities, we may extract the root of both. We shall then have

$$x+a=\pm\sqrt{b+a^2};$$

$$\therefore\ x=-a\pm\sqrt{b+a^2}.$$

The step previous to extracting the square root of both members is called completing the square, and the two may be enunciated in the following

RULE.

Reduce the equation to the above form, then add the square of half the coefficient of the second term to both members of the equation, and the first member will be a perfect square.

We perceive by the foregoing solution, that

The first value of the unknown quantity is equal to half the coefficient of the second term taken with a contrary sign, plus the square root of the absolute

term increased by the square of half the same coefficient.

The second value of the unknown quantity is equal to half the coefficient of the second term taken with a contrary sign, minus the square root of the absolute term increased by the square of the same coefficient.

Either of these values being substituted in the equation for the unknown quantity will verify it, that is, make both members equal. For we have from the first value

$$x^2=(-a+\sqrt{b+a^2})^2= \quad a^2-2a\sqrt{b+a^2}+b+a^2$$
$$2ax=2a\times(-a+\sqrt{b+a^2})=-2a^2+2a\sqrt{b+a^2};$$
$$\therefore\ x^2+2ax=b.$$

And for the second value we have

$$x^2=(-a-\sqrt{b+a^2})^2= \quad a^2+2a\sqrt{b+a^2}+b+a^2$$
$$2ax=2a(-a-\sqrt{b+a^2})=-2a^2-2a\sqrt{b+a^2};$$
$$\therefore\ x^2+2ax=b.$$

Hence the values found in the preceding solution are *roots* of the equation.

Examples.

1. Given $3x-\frac{3x-10}{9-2x}=2+\frac{6x^2-40}{2x-1}$ to find the values of x.

Clearing the equation of fractions and reducing, we have

$$x^2-\frac{31x}{2}=-46.$$

The equation is now of the proper form. Complete the square, and

$$x^2-\frac{31}{2}x+\frac{961}{16}=-46+\frac{961}{16}.$$

Extracting the root,

$$x-\frac{31}{4}=\pm\sqrt{-46+\frac{961}{16}}=\pm\frac{15}{4},$$

$$x=\frac{31}{4}=\pm\frac{15}{4};$$

$$\therefore\ x=11\tfrac{1}{2} \text{ or } 4.$$

Verification:

Substituting $11\frac{1}{2}$ for x in the original equation, gives

$$\frac{69}{2}-\frac{\frac{69}{2}-10}{9-23}=2+\frac{\frac{1587}{2}-40}{22},$$

or

$$\frac{69}{2}-\frac{\frac{49}{2}}{-14}=2+\frac{1507}{44},$$

or

$$\frac{69}{2}+\frac{49}{28}=2+\frac{137}{4}$$

$$\frac{1015}{28}=\frac{145}{4};$$

$$\therefore\ 0=0.$$

Again, substituting 4 for x in the original equation, gives

$$12-2=2+\frac{56}{7},$$

or

$$10=2+8.$$

2. Given $x^2-6x+8=80$ to find the values of x.

Ans. $x=12$ or -6.

3. Given $7x^2-21x+13=293$ to find the values of x.

Ans. $x=8$ or -5.

4. Given $\frac{2x^2}{3}+\frac{7}{2}-\frac{x}{2}=8$ to find the values of x.

Ans. $x=3$ or $-2\frac{1}{4}$.

5. Given $4x-\frac{36-x}{x}=46$ to find the values of x.

Ans. $x=12$ or $-\frac{3}{4}$.

6. Given $\frac{x+3}{2}+\frac{16-2x}{2x-5}=\frac{26}{5}$ to find the values of x.

Ans. $x=5$ or $6\frac{9}{10}$.

7. Given $14+4x-\frac{x+7}{x-7}=3x+\frac{9+4x}{3}$ to find the values of x.

Ans. $x=9$ or 28.

8. Given $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$ to find the values of x.

Ans. $x=2$ or -3.

9. Given $x^2-\frac{1617x}{21}=-\frac{20748}{21}$ to find the values of x.

Ans. $x=60.72$ or 16.27.

10. Given $\frac{x+11}{x}+\frac{9+4x}{x^2}=7$ to find the values of x.

Ans. $x=3$ or $-\frac{1}{2}$.

11. Given $x^2-\frac{11}{3}x=-2$ to find the values of x.

Ans. $x=3$ or $\frac{2}{3}$.

12. Given $x+\frac{7x-8}{x}=9$ to find the values of x.

Ans. $x=4$ or -2.

13. Given $acx^2+bcx-adx=bd$ to find the values of x.

Ans. $x=\frac{d}{c}$ or $-\frac{b}{a}$.

14. Given $5x^2-27x=162$ to find the values of x.

Ans. $x=9$ or $-3\frac{3}{5}$.

15. Given $\frac{1}{2}x^2-\frac{1}{3}x+20\frac{1}{2}=42\frac{2}{3}$ to find the values of x.

Ans. $x=7$ or $-\frac{19}{3}$.

16. Given $x^2-\frac{2}{5}x=-\frac{3}{25}$ to find the values of x.

Ans. $x=\frac{3}{5}$ or $-\frac{1}{5}$.

(111.) The method we have just been considering, of solving equations of the second degree, is equally applicable to all trinomial equations, in which the exponent of the unknown quantity in one term is double that in the other. For if we have the equation

$$x^{2n}+2ax^n=b,$$

and we put $y=x^n$, then $y^2=x^{2n}$, and we shall have

$$y^2+2ay=b.$$

Solving this by the preceding rule, gives

$$y=-a\pm\sqrt{b+a^2},$$

or

$$x^n=-a\pm\sqrt{b+a^2};$$

$$\therefore x=\sqrt[n]{-a\pm\sqrt{b+a^2}}.$$

EXAMPLES.

1. Given $2x^4-x^2+96=99$ to find the values of x.

By transposition and reduction, we have

$$x^4-\frac{1}{2}x^2=\frac{3}{2}.$$

Complete the square, and

$$x^4-\frac{1}{2}x^2+\frac{1}{16}=\frac{25}{16}.$$

Extracting the root,

$$x^2-\frac{1}{4}=\pm\frac{5}{4};$$

$$\therefore\ x^2=\frac{3}{2}\text{ or }-1;$$

$$\therefore\ x=\pm\sqrt{\frac{3}{2}}\text{ or }\pm\sqrt{-1}.$$

(112.) We may remark that *an equation of the second degree has two roots, of the third degree three roots, and of the* m*th degree* m *roots.*

2. Given $x^3-x^{\frac{3}{2}}=56$ to find the values of x.

By completing the square,

$$x^3-x^{\frac{3}{2}}+\frac{1}{4}=\frac{225}{4}.$$

And extracting the square root,

$$x^{\frac{3}{2}}-\frac{1}{2}=\pm\frac{15}{2}$$

$$x^{\frac{3}{2}}=\frac{1}{2}\pm\frac{15}{2}=8\text{ or }-7$$

$$x=\sqrt[3]{8^2}=4\text{ and }x=\sqrt[3]{(-7)^2}.$$

3. Given $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44$ to find the values of x.

Ans. $x=\pm 8$ or $\pm(-11)^{\frac{3}{2}}$.

4. Given $4x^{\frac{1}{3}}+x^{\frac{1}{6}}=39$ to find the values of x.

Ans. $x=729$ or $\left(\frac{13}{4}\right)^6$.

5. Given $3x^6+42x^3=3321$ to find the values of x.

Ans. $x=3$ or $-\sqrt[3]{41}$.

6. Given $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$ to find the values of x.

Ans. $x=243$ or $(-28)^{\frac{5}{3}}$.

7. Given $81x^4+17x^2=99x^2-1$ to find the values of x.

Ans. $x=1$, or -1, or $-\frac{1}{9}$.

8. Given $25x^2+6+\frac{4}{9x^2}=\frac{955}{9}$ to find the values of x.

Ans. $x=2$, or -2, or $-\frac{1}{15}$.

9. Given $x^{\frac{1}{2}}-5x^{\frac{1}{4}}=-6$ to find the values of x.

Ans. $x=16$ or 81.

10. Given $x-x^{\frac{1}{2}}=20$ to find the values of x.

Ans. $x=25$.

11. Given $x^{2n}-\frac{2}{3}x^n=\frac{8}{3}$ to find the value of x.

Ans. $x=\sqrt[n]{2}$.

12. Given $mx^2-2mx\sqrt{n}=nx^2-mn$ to find the values of x.

Ans. $x=\frac{\sqrt{mn}}{\sqrt{m}-\sqrt{n}}$ or $x=\frac{\sqrt{mn}}{\sqrt{m}+\sqrt{n}}$.

13. Given $abx^2+\frac{3a^2x}{c}=\frac{6a^2+ab-2b^2}{c^2}-\frac{b^2x}{c}$ to find the values of x.

Ans. $x=\frac{2a-b}{ac}$ or $x=-\frac{3a+2b}{bc}$.

(113.) It may also be observed that the preceding rule is equally applicable to all equations that are of the form

$$(x^{2n}+2ax^n)^{2n}+(x^{2n}+2ax^n)^n=b.$$

For we might put $x^{2n}+2ax^n=y$, and the equation would become $y^{2n}+y^n=b$, which is the same in form as that in *Article* 111. But we shall proceed to solve

it without such substitution. Let us resume the equation

$$(x^{2n}+2ax^n)^{2n}+(x^{2n}+2ax)^n=b.$$

Completing the square, we have

$$(x^{2n}+2ax^n)^{2n}+(x^{2n}+2ax)^n+\frac{1}{4}=b+\frac{1}{4},$$

and, extracting the square root,

$$(x^{2n}+2ax^n)^n+\frac{1}{2}=\pm\sqrt{b+\frac{1}{4}}.$$

Transposing, we obtain

$$(x^{2n}+2ax^n)^n=-\frac{1}{2}\pm\sqrt{b+\frac{1}{4}}.$$

Extracting the nth root, gives

$$x^{2n}+2ax^n=\sqrt[n]{-\frac{1}{2}\pm\sqrt{b+\frac{1}{4}}}.$$

(114.) *The* n*th root of a quantity is such a quantity as, being multiplied by itself* n−1 *times, will produce the given quantity.*

Hence we may put $\sqrt[n]{-\frac{1}{2}\pm\sqrt{b+\frac{1}{4}}}=c$; with this last substitution, the equation becomes

$$x^{2n}+2ax^n=c,$$

an equation which has been solved in *Art.* 111.

EXAMPLES.

1. Given $2(1+x-x^2)-\sqrt{1+x-x^2}=-\frac{1}{9}$ to find x.

Dividing both members by 2, we have

$$(1+x-x^2)-\tfrac{1}{2}(1+x-x^2)^{\frac{1}{2}}=-\frac{1}{18}.$$

Completing the square, gives

$$(1+x-x^2)-\tfrac{1}{2}(1+x-x^2)^{\frac{1}{2}}+\frac{1}{16}=\frac{1}{144}.$$

Extracting the square root,

$$(1+x-x^2)^{\frac{1}{2}}-\frac{1}{4}=\pm\frac{1}{12}$$

$$(1+x-x^2)^{\frac{1}{2}}=\frac{1}{4}\pm\frac{1}{12}=\frac{1}{3}\text{ or }\frac{1}{6};$$

$$\therefore\ 1+x-x^2=\frac{1}{9}\text{ or }\frac{1}{36}.$$

Transposing, and using the first value $\frac{1}{9}$, we have

$$x^2-x=\frac{8}{9},$$

which gives $x=\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{8}{9}}=\frac{1}{2}\pm\frac{1}{6}\sqrt{41}.$

And using the second value $\frac{1}{36}$, we have

$$x^2-x=\frac{35}{36};$$

$$\therefore\ x=\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{35}{36}}=\frac{1}{2}\pm\frac{1}{6}\sqrt{44}.$$

2. Given $(9x+4)+2(9x+4)^{\frac{1}{2}}=15$ to find the values of x. *Ans.* $x=\frac{5}{9}$ or $\frac{7}{3}$.

3. Given $(x+10)^{\frac{1}{2}}-(x+10)^{\frac{1}{4}}=2$ to find the value of x. *Ans.* $x=6$.

4. Given $(2x^2+3x+9)-5(2x^2+3x+9)^{\frac{1}{2}}=6$ to find the values of x.

Ans. $x=3$, or $-\frac{9}{2}$, or $x=\frac{1}{4}(-3\pm\sqrt{-55})$.

5. Given $(2x^2+1)+(2x^2+1)^{\frac{1}{2}}=12$ to find the values of x.

Ans. $x=2$ or -2, or $\frac{1}{2}\sqrt{30}$ or $-\frac{1}{2}\sqrt{30}$.

6. Given $(2x^2-4x+1)^2-(2x^2-4x+1)=42$ to find the values of x. *Ans.* $x=3$.

(115.) Equations of the higher degrees may be solved by the preceding rule, provided we can decompose them into polynomials that are similar to each other, the exponent of one of these polynomials being double that of the other.

1. For example, let us take the equation

$$x^{4n}-2x^{3n}+x^n=6,$$

by the method of extracting the square root of a polynomial.

$$\begin{array}{l|l}
x^{4n}-2x^{3n}+x^n & x^{2n}-x^n \\
\cline{2-2}
x^{4n} & 2x^{2n}-x^n \\
\hline
\quad -2x^{3n}+x^n & \quad -\ x^n \\
\quad -2x^{3n}+x^{2n} & \\
\cline{1-1}
\qquad -x^{2n}+x^n &
\end{array}$$

$$\therefore\ x^{4n}-2x^{3n}+x^n=(x^{2n}-x^n)^2-(x^{2n}-x^n).$$

We then have

$$(x^{2n}-x^n)^2-(x^{2n}-x^n)=6.$$

And, completing the square, we obtain

$$(x^{2n}-x^n)^2-(x^{2n}-x^n)+\frac{1}{4}=\frac{25}{4}.$$

Extracting the square root,

$$x^{2n}-x^n-\frac{1}{2}=\pm\frac{5}{2}$$

$$x^{2n}-x^n=3 \text{ or } -2.$$

Taking the first value, we have

$$x^{2n}-x^n=3.$$

Completing the square,

$$x^{2n}-x^n+\frac{1}{4}=\frac{13}{4}.$$

Extracting the square root,

$$x^n-\frac{1}{2}=\pm\frac{1}{2}\sqrt{13}$$

$$x^n=\frac{1}{2}\pm\frac{1}{2}\sqrt{13};$$

$$\therefore\ x=\sqrt[n]{\frac{1}{2}\pm\frac{1}{2}\sqrt{13}}.$$

2. Given $x^3-4x^2+6x=4$ to find the values of x.

Multiply both members of the equation by x, and we have

$$x^4-4x^3+6x^2=4x,$$

or, by transposing,

$$x^4-4x^3+6x^2-4x=0.$$

This can be decomposed into the polynomials

$$(x^2-2x)^2+2(x^2-2x)=0.$$

By completing the square and extracting the root,

$$x^2-2x+1=\pm1$$

$$x-1=\pm\sqrt{\pm1}.$$

The three roots of the proposed equation are 1, $1+\sqrt{-1}$, and $1-\sqrt{-1}$. The value of x, which is equal to $1-1$ or 0, belongs to the equation of the fourth degree, obtained by multiplying the given equation by x.

3. Given $x^4-10x^3+35x^2-50x+24=0$ to find the values of x. *Ans.* $x=1$, 2, 3, or 4.

4. Given $x^4-2x^3+x=132$ to find the values of x.

Ans. $x=4$ or -3.

(116.) Familiarity with such transformations, and a facility in applying them, enable the analyst to

push his researches into fields of inquiry which would otherwise be beyond his reach, since his symbols would become so complex that he could neither operate upon nor interpret them.

Equations of the form

$$\sqrt{x^2+ax+b}+\sqrt{x^2-ax+b}=c \quad . \quad . \quad (1)$$

are frequently met with in Geometry. The usual method of freeing the equation from radicals would be very complicated; but the analyst seizes the truth, which the form of the equation presents to him,

$$\{\sqrt{x^2+ax+b}+\sqrt{x^2-ax+b}\}\times\{\sqrt{x^2+ax-b} \\ -\sqrt{x^2-ax+b}\}=2ax \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

then $$\sqrt{x^2+ax+b}-\sqrt{x^2-ax+b}=\frac{2ax}{c} \quad . \quad (3)$$

by dividing the first member by the first factor, and the second member by the value of the first factor.

If we now add the first and third equations together, member by member, we shall have

$$2\sqrt{x^2+ax+b}=\frac{2ax}{c}+c,$$

or $$\sqrt{x^2+ax+b}=\frac{ax}{c}+\tfrac{1}{2}c.$$

Squaring both members, we obtain

$$x^2+ax+b=\frac{a^2x^2}{c^2}+ax+\tfrac{1}{4}c^2$$

$$x^2-\frac{a^2x^2}{c^2}=\tfrac{1}{4}c^2-b,$$

$$\left(1-\frac{a^2}{c^2}\right)x^2=\tfrac{1}{4}c^2-b$$

$$x=\pm\frac{c}{2}\sqrt{\frac{c^2-4b}{c^2-a^2}}.$$

(117.) DISCUSSION OF EQUATIONS OF THE SECOND DEGREE.

We have seen (*Art.* 110) that every complete equation of the second degree can be reduced to the form

$$x^2+2ax=b \quad . \quad . \quad . \quad . \quad (1)$$

in which a and b are any numbers whatever, either whole or fractional, positive or negative.

The value of x in this equation is

$$x=-a+\sqrt{b+a^2} \quad . \quad . \quad . \quad . \quad (2)$$

or

$$x=-a-\sqrt{b+a^2} \quad . \quad . \quad . \quad . \quad (3)$$

And, therefore,

Every equation of the second degree has two roots, and only two.

If we take the algebraic sum of the two roots, we shall find it equal to -2a, *or the coefficient of the second term taken with a contrary sign. If the two roots are numerically equal, with opposite signs, their sum is zero, and the second term vanishes.*

Transpose the second members of (2) and (3), we shall have

$$x+a-\sqrt{b+a^2}=0 \quad . \quad . \quad . \quad (4)$$

$$x+a+\sqrt{b+a^2}=0 \quad . \quad . \quad . \quad (5)$$

Multiplying equations (4) and (5) together, we obtain

$$x^2+2ax-b,$$

the original equation. Hence (4) and (5) are the factors into which a complete equation is decomposed.

Hence, *every equation of the second degree may be resolved into two factors.*

Thus, the equation $x^2+8x=20$, being solved, gives

$$x=2 \text{ and } x=-10;$$

$$\therefore\ (x-2)(x+10)=x^2+8x-20.$$

It is perfectly evident, that if r is a root of an equation of the second degree, that equation will be divisible by $x-r$. The preceding equation gives 2 as a root, hence it must be divisible by $x-2$.

1. What are the factors, and what is the equation, of which the roots are a and $-b$?

Here the factors are

$$x-a \text{ and } x+b.$$

Consequently,

$$(x-a)(x+b)=x^2+(b-a)x-ab,$$

or

$$x^2+(b-a)x=ab,$$

is the equation required.

2. What are the factors, and what is the equation, of which the roots are -9 and 3?

Ans. The binomial factors are $x+9$ and $x-3$, and the equation is $x^2+6x=27$.

3. What are the factors, and what is the equation, of which the roots are 10 and -4?

Ans. $(x-10)(x+4)$ are the factors, and $x^2-6x=40$ is the equation.

4. What are the factors, and what is the equation, of which the roots are

$$\frac{7+\sqrt{-1039}}{16} \text{ and } \frac{7-\sqrt{-1039}}{16}?$$

Ans. $x-\dfrac{7+\sqrt{-1039}}{16}$ and $x-\dfrac{7-\sqrt{-1039}}{16}$

$$8x^2-7x=-34.$$

If we multiply the second members of equations (2) and (3) together, we shall have

$$(-a+\sqrt{b+a^2})\times(-a-\sqrt{b+a^2})=-b;$$

that is, *the product of the two roots of an equation of the second degree is equal to the absolute term taken with an opposite sign.*

(118.) In the general form of the equation, let $2a$ be positive, b also positive, and less than a^2.

If $2a$ be positive,

$x=-a+\sqrt{b+a^2}$ the value of x is positive,

$x=-a-\sqrt{b+a^2}$ " " " negative.

If $2a$ be negative,

$x=+a+\sqrt{b+a^2}$ the value of x is positive,

$x=+a-\sqrt{b+a^2}$ " " " negative.

If b be greater than a^2, then,

If $2a$ be positive,

$x=-a+\sqrt{b+a^2}$ the value of x is positive,

$x=-a-\sqrt{b+a^2}$ " " " negative.

If $2a$ be negative,

$x=+a+\sqrt{b+a^2}$ the value of x is positive,

$x=+a-\sqrt{b+a^2}$ " " " negative.

If b is negative, $2a$ positive, and $b<a^2$, then

$$\left.\begin{array}{l} x=-a+\sqrt{-b+a^2} \\ x=-a-\sqrt{-b+a^2} \end{array}\right\} \text{both negative.}$$

If $2a$ is negative, then

$$\left.\begin{array}{l} x=a+\sqrt{-b+a^2} \\ x=a-\sqrt{-b+a^2} \end{array}\right\} \text{both positive.}$$

Let b be negative, and greater than a^2; then,

If $2a$ is positive,

$$\left.\begin{array}{l} x=-a+\sqrt{-b+a^2} \\ x=-a-\sqrt{-b+a^2} \end{array}\right\} \text{both imaginary.}$$

If $b=a^2$, and be negative, then,

If $2a$ is positive, $x=-a$ } and the two values are
$2a$ is negative, $x=\ \ a$ } equal.

If $b=0$, then, when

$2a$ is positive, $x=-2a$ or $=0$,

$2a$ is negative, $x=\ \ 2a$ or $=0$.

If b is positive, and $2a=0$, then

$$x=\pm\sqrt{b},$$

the two values are equal, with contrary signs.

If b is negative, and $2a=0$, then

$$x=\pm\sqrt{-b},$$

and both values are imaginary.

If $b=0$, and $2a=0$, then both values of x are *zero*.

One case, which is frequently met with in the solution of problems, deserves a passing notice; that is,

Given $$ax^2+bx=c;$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

Suppose, now, that $a=0$, then

$$x=\frac{-b\pm b}{0};$$

$$\therefore\ x=\frac{0}{0},$$

or $$x=\frac{-2b}{0}.$$

The first value is of the indeterminate form, and the second the symbol of infinity.

Let us interpret the first.

If, in the original equation, we put $x=\frac{1}{x'}$, we shall have

$$\frac{a}{x'^2}+\frac{b}{x'}=c,$$

and, clearing the equations of fractions,

$$a+bx'=cx'^2,$$

or $$cx'^2-bx'=a.$$

And putting $a=0$, it becomes, by dividing x', which is $=0$,

$$cx'=b;$$

$$\therefore\ x'=\frac{b}{c}.$$

Hence $$x=\frac{1}{x'}=\frac{c}{b};$$

$$\therefore\ x=\frac{1}{0} \text{ and } x=\frac{c}{b}$$

are the two values of x.

We here perceive that the first value

$$x=\frac{0}{0}=\frac{c}{b}.$$

In regard to the second, viz.,

$$x=\frac{-2b}{0}.$$

If, in the original equation, we make $a=0$, then

$$bx=c,$$

which is an equation of the first degree, and has but *one root*.

If we ask an *absurd* question, the language of Algebra gives us an equivalent answer. Hence we may understand that, *whenever we obtain an imaginary expression, or a symbol of infinity, as the value of the unknown quantity, conditions have been imposed that are impossible to be fulfilled.*

For example, suppose we had the problem:

Divide the number 20 into two such parts that their product shall be equal to 150.

If we put $x=$ one of the parts,

then $20-x=$ the other,

and $20x-x^2=150$,

or $x^2-20x=-150$;

$$\therefore\ x=10\pm\sqrt{-50}.$$

An imaginary expression. We have asked an impossibility, and have received an answer in the same terms.

(119.) It is a proposition easily demonstrated, that *if a given number be divided into two parts, and those parts be multiplied together, their product will be the greatest possible when the two parts are equal.*

Let n be any number whatever, and d the difference of the two parts into which it is to be divided.

Then $\frac{n}{2}+\frac{d}{2}=$ the greater part,

and $\frac{n}{2}-\frac{d}{2}=$ the less part.

Then their product will be

$$\left(\frac{n}{2}+\frac{d}{2}\right)\left(\frac{n}{2}-\frac{d}{2}\right)=\frac{n^2}{4}-\frac{d^2}{4}.$$

It is perfectly evident that the smaller d becomes, the greater will be the value of $\frac{n^2}{4}-\frac{d^2}{4}$, so that when $d=0$, the greatest product becomes

$$\frac{n^2}{4}=\frac{n}{2}\times\frac{n}{2};$$

or, when the two parts are equal, applying this to the problem above, we shall find that 20 can be divided into two parts, whose product shall be equal to 100, but equal to no *greater* quantity.

(120.) The solution of trinomial equations of the

fourth degree requires sometimes the extraction of the square root of a quantity of the form

$$a \pm \sqrt{b},$$

in which a and b are numerical or algebraic quantities.

We can assume

$$\sqrt{a+\sqrt{b}}=p+q \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\sqrt{a-\sqrt{b}}=p-q \quad . \quad . \quad . \quad . \quad (2)$$

where p and q are quantities to be determined.

Multiply these equations together, member by member, and we shall have

$$\sqrt{a^2-b}=p^2-q^2 \quad . \quad . \quad . \quad . \quad (3)$$

Now a^2-b must be a perfect square, in order that p^2-q^2 be a rational quantity. If a^2-b is not a perfect square, the transformation is not used.

Put $\quad \therefore p^2-q^2=c=\sqrt{a^2-b} \quad . \quad . \quad . \quad (4)$

Square (1) and (2), we have

$$a+\sqrt{b}=p^2+2pq+q^2$$

$$a-\sqrt{b}=p^2-2pq+q^2.$$

Add these member to member, and we obtain

$$p^2+q^2=a \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Adding (4) and (5) gives

$$2p^2=a+c.$$

Subtracting (4) from (5) gives

$$2q^2=a-c;$$

$$\therefore p=\pm\sqrt{\frac{a+c}{2}} \text{ and } q=\pm\sqrt{\frac{a-c}{2}}.$$

Hence $\sqrt{a+\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}\right) \quad . \quad (6)$

and $\quad \sqrt{a-\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}\right) \quad . \quad (7)$

To extract the square root of a binomial of the form $a \pm \sqrt{b}$, we have the following

Rule.

From the square of the first term take the square of the second, extract the square root of the remainder, and, denoting that root by c, *the root required will be*

$$\sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}.$$

Examples.

1. Required the square root of $3+2\sqrt{2}$.

Here $\qquad a=3$, and $\sqrt{b}=2\sqrt{2}$,

$$\sqrt{a^2-b}=\sqrt{9-8}=1=c;$$

$$\therefore \sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}=\sqrt{\frac{3+1}{2}}+\sqrt{\frac{3-1}{2}};$$

$$\therefore \sqrt{3+2\sqrt{2}}=\sqrt{2}+1, \textit{ Ans.}$$

Verification:

$$(\sqrt{2}+1)^2=2+2\sqrt{2}+1=3+2\sqrt{2}.$$

2. Required the square root of $19+8\sqrt{3}$, or find the value of $\sqrt{19+8\sqrt{3}}$.

Ans. $4+\sqrt{3}$.

3. Required the square root of $12-\sqrt{140}$.

Ans. $\sqrt{7}-\sqrt{5}$.

4. Required the square root of $7-2\sqrt{10}$.

Ans. $\sqrt{5}-\sqrt{2}$.

5. Required the square root of $7+4\sqrt{3}$, or reduce $\sqrt{7+4\sqrt{3}}$ to its simplest form.

Ans. $2+\sqrt{3}$.

6. Reduce $\sqrt{-1+4\sqrt{-5}}$ to its simplest form.

Here $a=-1$, and $\sqrt{b}=4\sqrt{-5}$;

$$\therefore\ a^2=1, \text{ and } b=16\times(-5)=-80,$$

$$\sqrt{a^2-b}=\sqrt{1-(-80)}=\sqrt{81}=9=c;$$

$$\therefore \sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}=\sqrt{\frac{-1+9}{2}}+\sqrt{\frac{-1-9}{2}}$$

$$=2+\sqrt{-5},\ \textit{Ans.}$$

7. Reduce $\sqrt{31+12\sqrt{-5}}$ to its simplest form.

Ans. $6+\sqrt{-5}$.

8. Reduce $\sqrt{18-10\sqrt{-7}}$ to its simplest form.

Ans. $5-\sqrt{-7}$.

9. Reduce $\sqrt{-1-4\sqrt{-3}}$ to its simplest form.

Ans. $2-\sqrt{-3}$.

10. Reduce $\sqrt{14+6\sqrt{5}}$ to its simplest form.

Ans. $3+\sqrt{5}$.

(121.) Problems producing Equations of the Second Degree containing but one Unknown Quantity.

1. Given the sum of two numbers equal 6, and the sum of their fourth powers 272, to find the numbers.

Let $3+x$=greater, and $3-x$=less.

Then $(3+x)^4+(3-x)^4=272$.

Expanding and reducing, we have

$$x^4+54x^2=55.$$

Completing the square,

$$x^4+54x^2+729=784.$$

Extracting the square root,

$$x^2+27=\pm 28$$

$$x^2=-27\pm 28$$

$$x=\pm 1 \text{ or } \pm\sqrt{-55}.$$

The negative value will not answer the problem;

$$\therefore\ 3+x=4=\text{greater},$$

and $$3-x=2=\text{less}.$$

2. A person at play won, at the first game, as much money as he had in his pocket; at the second, he won 5 shillings more than the square root of what he then had; at the third game, he won the square of all he then had, and found that he possessed £112 16*s*. What had he at first?

Let x=the number of shillings he had at first.

Then $2x$=amount after the first game,

and $\sqrt{2x}+5$=number won the second game;

$\therefore\ 2x+\sqrt{2x}+5$=amount after the second game,

and $(2x+\sqrt{2x}+5)^2$=number won the third game;

$$\therefore\ (2x+\sqrt{2x}+5)^2+(2x+\sqrt{2x}+5)=2256.$$

Complete the square, and we have

$$(2x+\sqrt{2x}+5)^2+(2x+\sqrt{2x}+5)+\frac{1}{4}=\frac{9025}{4}.$$

Extracting the square root, gives

$$2x+\sqrt{2x}+5+\frac{1}{2}=\pm\frac{95}{2};$$

$$\therefore\ 2x+\sqrt{2x}=42 \text{ or } -53.$$

Using the positive value, we have

$$2x+\sqrt{2x}=42.$$

Completing the square,

$$2x+\sqrt{2x}+\frac{1}{4}=\frac{169}{9}$$

$$\sqrt{2x}+\frac{1}{2}=\pm\frac{13}{2}$$

$$\sqrt{2x}=6 \text{ or } -7;$$

$$\therefore\ 2x=36 \text{ or } 49;$$

$$\therefore\ x=18 \text{ shillings, } \textit{Ans.}$$

3. Two persons, A and B, started at the same time for the same place, distant 150 miles. A traveled 3 miles an hour faster than B, and arrived at the end of his journey 8 hours and 20 minutes before him. At what rate did each person travel per hour?

Ans. A 9 miles an hour,
B 6 miles.

4. There are two numbers whose product is 120; if 2 be added to the less, and 3 subtracted from the greater, the product of the sum and difference will also be 120. Required the numbers.

Ans. 15 and 8.

5. Find a number from the cube of which, if 19 be subtracted, and the remainder multiplied by that cube, the product shall be 216.

Ans. 3 or -2.

6. Divide 20 into two such parts that the product of the whole and one of the parts shall be equal to the square of the other.

Ans. $10\sqrt{5}-10$ and $30-10\sqrt{5}$.

7. A merchant sold a quantity of brandy for £39, and gained as much per cent. as the brandy cost him. What was the price of the brandy?

Ans. £30.

8. A and B set out at the same time from two towns, which were at the distance of 247 miles, and traveled the direct road till they met. A went 9 miles a day; and the number of days, at the end of which they met, was greater by 3 than the number of miles B went in a day. How many miles did each go?

Ans. A 117 miles,
B 130 miles.

9. A and B purchased a farm containing 900 acres, at the rate of \$2 per acre, which they paid equally between them; but, on dividing the same, A got that part of the farm which contained the best of the improvements, and agreed to pay 45 cents per acre more than B. How many acres had each, and at what price per acre?

Put x=number of acres A had;

then $900-x$= " " B "

and $\frac{900}{x}$=price per acre paid by A,

$$\frac{900}{900-x}+\frac{45}{100}=\frac{900}{x}.$$

Ans. A=400 acres at \$2.28,
B=500 acres at \$1.80.

10. A company at a tavern had £8 15*s.* to pay, but before the bill was settled two of them left, when those who remained had each 10 shillings more to pay. How many were there in the company at first?

Ans. 7.

11. Bought two flocks of sheep for £65 13*s.*, one containing 5 more than the other. Each sheep cost as many shillings as there were sheep in the flock. What was the number in each flock?

Ans. 23 and 28.

12. A laborer dug two ditches, one of which was 6 yards longer than the other, for £17 16*s.*, and the digging of each cost as many shillings a yard as there were yards in its length. What was the length of each?

Ans. 10 and 16 yards.

13. In a parcel which contains 24 coins of silver and

copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each?

Ans. 6 of copper and 18 of silver.

14. Two messengers, A and B, were sent at the same time to a place 93 miles distant; the former of whom, riding one mile an hour more than the latter, arrived at the end of his journey an hour before him. At what rate did each travel per hour?

Ans. A went 10 miles an hour,
B went 9 miles an hour.

15. The sum of two numbers is 8, and the sum of their cubes 152. What are the numbers?

Ans. 3 and 5.

16. The sum of two numbers is 8, and the sum of their fourth powers 706. What are the numbers?

Ans. 3 and 5.

17. A sets out from C toward D, and travels 8 miles a day. After he had gone 27 miles, B set out from D toward C, and goes every day $\frac{1}{20}$th of the whole journey, and after he had traveled as many days as he goes miles in one day, he met A. Required the distance of the place C from D.

Ans. 180 or 60 miles.

18. A person bought a number of oxen for $80, and if he had bought 4 more for the same sum he would have paid one dollar less for each. How many did he buy?

Ans. 16.

Interpret the negative value -20.

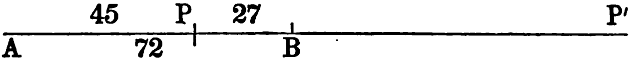
19. Two church bells, whose loudness of tone are as p to q, are a miles apart. Now, supposing the

strength of sound to be inversely as the square of the distance, at what point between the two will the bells be equally heard?

Ans. $\dfrac{a\sqrt{p}}{\sqrt{p}\pm\sqrt{q}}$ miles from the first,

and $\dfrac{a\sqrt{q}}{\sqrt{p}\pm\sqrt{q}}$ " " second.

20. Two lights, whose intensities are as 25 to 9, are placed at the distance of 72 inches from each other. Find, on the line which joins them, the point which will be equally illuminated by each, admitting that the intensity of light varies inversely as the square of the distance.



Ans. 45 inches from the large light and 27 from less light, or 180 from large and 108 from smaller.

(122.) Equations of the Second Degree involving two or more Unknown Quantities.

An equation containing two or more unknown quantities is of the *second degree* when *the greatest sum of the exponents of the unknown quantities in any term is equal to* 2.

The solution of equations of the second degree may sometimes be simplified by particular artifices, which we shall illustrate in the following

Examples.

(123.) *When the equations are homogeneous, their solution may be effected by assuming* $x=vy$.

1. Given $\left.\begin{array}{r}2x^2+3xy+y^2=20\\ 5x^2+4y^2=41\end{array}\right\}$ to find x and y.

Let $x=vy$, then the equations become

$$2v^2y^2+3vy^2+y^2=20 \therefore y^2=\frac{20}{2v^2+3v+1},$$

and $$5v^2y^2+4y^2=41 \therefore y^2=\frac{41}{5v^2+4}.$$

Equate these two values of y^2, and we have

$$\frac{20}{2v^2+3v+1}=\frac{41}{5v^2+4}.$$

Clearing of fractions, it becomes

$$100v^2+80=82v^2+123v+41.$$

Transposing and reducing, gives

$$v^2-\frac{41}{6}v=-\frac{13}{6},$$

from which v may be obtained; thence y and x. There will be four values, both of x and y, which satisfy the equations.

2. Given $\left.\begin{array}{r}4x^2-2xy=12\\ 3xy+2y^2=\ \ 8\end{array}\right\}$ to find x and y.

Ans. $x=+2$ or -2,
$y=1$ or -1.

3. Given $\left.\begin{array}{r}x^2+xy=56\\ xy+2y^2=60\end{array}\right\}$ to find x and y.

Ans. $x=\pm4\sqrt{2}$ or ∓14,
$y=\pm3\sqrt{2}$ or ±10.

(124.) *When the unknown quantities in each equation are similarly involved, the operation may sometimes be shortened by substituting for the unknown quantities the sum and difference of two others.*

4. Given $\left.\begin{array}{r}x+y=2a\\ x^5+y^5=\ b\end{array}\right\}$ to find x and y.

Put $\qquad x=a+z$, and $y=a-z$;

then $\qquad x+y=2a$,

and $\qquad (a+z)^5+(a-z)^5=b$.

Involving and adding, we have

$$2a^5+20a^3z^2+10az^4=b,$$

or $$z^4+2a^2z^2=\frac{b-2a^5}{10a};$$

$$\therefore\ z=\pm\sqrt{\left\{-a^2\pm\sqrt{\frac{b+8a^5}{10a}}\right\}}$$

and $$x=a\pm\sqrt{\left\{-a^2\pm\sqrt{\frac{b+8a^5}{10a}}\right\}}$$

and $$y=a\mp\sqrt{\left\{-a^2\pm\sqrt{\frac{b+8a^5}{10a}}\right\}}$$

5. Given $\left.\begin{array}{l} x+y=6 \\ x^5+y^5=1056 \end{array}\right\}$ to find x and y.

Ans. $x=4$ or 2, or $3\pm\sqrt{-19}$,
$y=2$ or 4, or $3\mp\sqrt{-19}$.

6. Given $\left.\begin{array}{l} x^3+y^3=18xy \\ x+y=12 \end{array}\right\}$ to find x and y.

Ans. $x=8$ or 4,
$y=4$ or 8.

(125.) *When two unknown quantities enter symmetrically into the equations, the solution is often simplified by substituting for them two other unknown quantities, one of which is their sum, the other their product.*

7. Given $\left.\begin{array}{l} x+y=5 \\ x^5+y^5=275 \end{array}\right\}$ to find x and y.

Put $\quad xy=p$.

$$(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$
$$=x^5+y^5+5xy(x^3+y^3)+10x^2y^2(x+y),$$

and $x^3+y^3=(x+y)^3-(3x^2y+3xy^2)$
$=(x+y)^3-3xy(x+y)$
$=125-15p.$

Hence $(x+y)^5=275+5p(125-15p)+50p^2$,

or $3125=275+625p-75p^2+50p^2$,

or $25p^2-625p=-2850$

$p^2-25p=-114.$

From which $p=19$ or 6;

$\therefore\ x=2$ or 3, or $\frac{1}{2}(5\pm\sqrt{-51})$

$y=3$ or 2, or $\frac{1}{2}(5\mp\sqrt{-51})$.

8. Given $\left.\begin{array}{l} x^3-x^2y-xy^2+y^3=7 \\ x^3+x^2y+xy^2+y^3=175 \end{array}\right\}$ to find x and y.

Put $x+y=s$ and $xy=p$.

Ans. $x=3$ or 4,
$y=4$ or 3.

9. Given $\left.\begin{array}{l} x^3+y^3=189 \\ x^2y+xy^2=180 \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$ or 5,
$y=5$ or 4.

10. Given $\left.\begin{array}{l} x^2+y^2+x+y=12 \\ x+y-xy=0 \end{array}\right\}$ to find x and y.

Ans. $x=2$ or $\frac{1}{2}(-3\pm\sqrt{21})$,
$y=2$ or $\frac{1}{2}(-3\mp\sqrt{21})$.

11. Given $\left.\begin{array}{l} x^2y+xy^2=6 \\ x^3y^2+x^2y^3=12 \end{array}\right\}$ to find x and y.

Ans. $x=1$ or 2,
$y=2$ or 1.

12. Given $\left.\begin{array}{l} x^2y^2+4xy=96 \\ x+y=6 \end{array}\right\}$ to find x and y.

Ans. $x=2$ or 4, or $3\pm\sqrt{21}$,
$y=4$ or 2, or $3\mp\sqrt{21}$.

13. Given $\left.\begin{array}{l} x^2+xy=12 \\ xy-2y^2=1 \end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 3$,
$y=\pm 1$.

14. Given $\left.\begin{array}{l} \sqrt[3]{x}+\sqrt[3]{y}=3 \\ x+y \quad =9 \end{array}\right\}$ to find x and y.

Ans. $x=8$ or 1,
$y=1$ or 8.

15. Given $\left.\begin{array}{l} x(\sqrt{y}+1)+2\sqrt{xy}=55-y(\sqrt{x}+1) \\ x\sqrt{y}+y\sqrt{x}=30 \end{array}\right\}$ to find x and y.

Ans. $x=9$ or 4,
$y=4$ or 9.

16. Given $\left.\begin{array}{l} xy+\dfrac{y^2}{x}=40 \\ \dfrac{x^2}{y}-xy=96 \end{array}\right\}$ to find x and y.

Ans. $x=\pm 8$,
$y=\pm 4$.

17. Given $\left.\begin{array}{l} x^4-y^4 \quad =1280 \\ x^2y+xy^2=480 \end{array}\right\}$ to find x and y.

Ans. $x=6$,
$y=2$.

18. Given $\left.\begin{array}{l} x^n(x^n+y^n+z^n+v^n)=a \\ y^n(x^n+y^n+z^n+v^n)=b \\ z^n(x^n+y^n+z^n+v^n)=c \\ v^n(x^n+y^n+z^n+v^n)=d \end{array}\right\}$ to find x, y, z, and v.

If we divide the first, second, and third, respectively by the fourth, we shall have

$$\frac{x^n}{v^n}=\frac{a}{d} \therefore x^n=\frac{av^n}{d}$$

$$\frac{y^n}{v^n}=\frac{b}{d} \therefore y^n=\frac{bv^n}{d}$$

$$\frac{z^n}{v^n}=\frac{c}{d} \therefore z^n=\frac{cv^n}{d}.$$

If we substitute these values in the first equation, and reduce, we shall find

$$(a+b+c+d)v^n=d^n \therefore v=\left\{\frac{d}{a+b+c+d}\right\}^{\frac{1}{n}}.$$

Hence $x=\left\{\dfrac{a}{a+b+c+d}\right\}^{\frac{1}{n}}$, $y=\left\{\dfrac{b}{a+b+c+d}\right\}^{\frac{1}{n}}$.

and $\quad z=\left\{\dfrac{c}{a+b+c+d}\right\}^{\frac{1}{n}}$.

19. Given $\left.\begin{array}{r}(x^2-y)^2=49\\(x^2-y^2)^2-(x^2-y^2)=20\end{array}\right\}$ to find x and y.

Ans. $x=\pm3$ or $\pm\sqrt{6}$,
$y=2$ or -1.

There is one value more of x, and one of y, which are left to be determined by the student.

20. Given $\left.\begin{array}{l}x^2y^4-7xy^2=1710\\xy-y=12\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$ or $\frac{1}{5}$,
$y=3$ or -15.

21. Given $\left.\begin{array}{l}x+y+\sqrt{x+y}=6\\x^2+y^2=10\end{array}\right\}$ to find x and y.

Ans. $x=3$ or 1, or $4\frac{1}{2}\pm\frac{1}{2}\sqrt{-61}$,
$y=1$ or 3, or $4\frac{1}{2}\mp\frac{1}{2}\sqrt{-61}$

22. Given $\left.\begin{array}{l}(x+y)^2-3y=28+3x\\ 2xy+3x=35\end{array}\right\}$ to find x and y.

Ans. $x=5$ or $\frac{7}{2}$, or $-\frac{5}{4}\pm\frac{1}{4}\sqrt{-255}$,

$y=2$ or $\frac{7}{2}$, or $-\frac{11}{4}\mp\frac{1}{4}\sqrt{-255}$.

23. Given $\left.\begin{array}{l}x^2+3x+y=73-2xy\\ y^2+3y+x=44\end{array}\right\}$ to find x and y.

Ans. $x=4$ or 16,

$y=5$ or -7.

24. Given $\left.\begin{array}{l}\frac{x^4}{y^2}+\frac{y^4}{x^2}=\frac{1225}{9}-2xy\\ x+y=10\end{array}\right\}$ to find x and y.

Ans. $x=6$ or 4,

$y=4$ or 6.

25. Given $\left.\begin{array}{l}y^4-12xy^2=432\\ y^2-\ 2xy=12\end{array}\right\}$ to find x and y.

Ans. $x=2$ or 3,

$y=6$ or $3+\sqrt{21}$.

(126.) Problems producing Equations of the Second Degree.

We may remark that, in the solution of problems which involve trinomial equations or quadratics, we sometimes obtain answers which do not correspond with the conditions. The reason is, that the algebraic language is more general than the common language, and the equation, which is a proper representation of the conditions, expresses also other conditions, and other suppositions than those of the problem.

1. Find two numbers such that the square of the

greater minus the square of the less may be 56, and the square of the less plus one third their product may be 40.

Let $\qquad x=$greater

$\qquad y=$less;

then, by the question,

$$x^2-y^2 \quad =56$$
$$y^2+\tfrac{1}{3}xy=40.$$

If we put $x=vy$, these equations will become

$$v^2y^2-y^2 \quad =56 \therefore y^2=\frac{56}{v^2-1}$$

$$y^2+\tfrac{1}{3}vy^2=40 \therefore y^2=\frac{40}{1+\frac{1}{3}v}.$$

Equate these two values of y^2, and we shall have

$$\frac{40}{1+\frac{1}{3}v}=\frac{56}{v^2-1}.$$

Clearing the equation of fractions and reducing, this gives

$$v=\frac{9}{5} \text{ or } -\frac{4}{3}.$$

From which we obtain

$$y=5 \text{ or } -5, \text{ or } \pm 6\sqrt{2}=\text{less},$$

and $\qquad x=9$ or $\mp 8\sqrt{2}$.

2. Find two numbers such that their difference shall be 98, and the difference of their cube roots 2.

Ans. 125 and 27.

3. The product of two numbers, multiplied by the sum of their squares, is 510, and the difference of their squares is 16. What are the numbers?

Put $x+y=$greater, and $x-y=$the less.

Ans. 5 and 3.

4. Given the sum of the cubes of two numbers 35,

and the sum of their ninth powers 20195, to find the numbers. *Ans.* 3 and 2.

5. Find two numbers such that their difference may be equal to 4, and their product, multiplied by the sum of their squares, may be 480.

Ans. 2 and 6.

6. Divide 140 into two such parts that the greater being divided by the less and the less by the greater, and the less quotient being multiplied by 8 and the greater by $4\frac{1}{2}$, the two products shall be equal.

Ans. 80 and 60.

7. Find two numbers such that their sum, product, and difference of their squares shall all be equal to each other.

Ans. $\frac{3}{2}+\frac{1}{2}\sqrt{5}$ and $\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

8. Find two numbers such that their sum shall be 13, and the sum of their fourth powers 4721.

Ans. 8 and 5.

9. Divide the number 40 into two such parts that the sum of their squares shall be 818.

Ans. 23 and 17.

10. Find a number such that if you subtract it from 10, and then multiply it by the number itself, the product shall be 21.

Ans. 7 or 3.

11. A farmer bought as many sheep as cost him \$1200, and, after reserving 15 out of the number, sold the remainder for \$1080, and gained \$2 a head by them. How many sheep did he buy, and at what price?

Ans. 175 sheep, \$16 a head.

12. A person ordered \$144 to be distributed among some poor people; but, before the money was divided, there came in two claimants more, by which means the former received one dollar a piece less than they would otherwise have done. What was the number at first?

Ans. 16.

13. A person being asked his age, answered, "My mother was 20 years old when I was born, and her age, multiplied by mine, exceeds our united ages by 2500." What was his age?

Ans. 42 years.

14. Two locomotives commence running at the same time from the two extremities of a rail-road of 324 miles in length, one traveling 3 miles an hour faster than the other, and they meet after having traveled as many hours as the slower traveled miles per hour. Find the distance each goes.

Ans. The faster 180 miles,
The slower 144 miles.

CHAPTER VI.

PROGRESSIONS.

SECTION I.

ARITHMETICAL PROGRESSION, OR PROGRESSION BY DIFFERENCES.

(127.) *An Arithmetical Progression*, or, more appropriately, *a progression by differences*, is a series of terms or quantities which increase or decrease continually by the addition or subtraction of a constant quantity. This constant quantity is called the *common difference*.

If we put a=the first term,
l=the last term,
d=the common difference,
n=the number of terms,
S=the sum of all the terms,

we shall have, if the series is an *increasing* one, the successive terms

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ \&c.,$$

so that the last or nth term is evidently

$$l=a+(n-1)d \quad . \quad . \quad . \quad . \quad (1)$$

But if the series be decreasing, the last term will be

$$l=a-(n-1)d.$$

(128.) *To find the sum of* n *terms of a series in arithmetical progression.*

We have $\quad S=a+(a+d)+(a+2d)+\ .\ .\ .\ l.$

Reversing the series $\quad S=l+(l-d)+(l-2d)+\ .\ .\ .\ a.$

Adding, $\quad 2S=(a+l)+(a+l)+(a+l)+\ .\ (a+l),$

to n terms, $\quad \therefore\ S=\frac{(a+l)n}{2}\ .\ .\ .\ (2)$

That is, *the sum of the terms of an arithmetical progression is equal to half the sum of the two extremes multiplied by the number of terms.*

From the equations (1) and (2), either two of the quantities a, l, d, n, and S can be determined when the other three are known.

Examples.

1. Find the sum of 60 terms of the series 5, 15, 25, 35, &c.

Here $\quad a=5,\ d=10,\ n=60;$

$$\therefore\ l=a+(n-1)d=5+59\times10=595.$$

Hence $\quad S=\frac{(a+l)n}{2}=\frac{(5+595)\times60}{2};$

$$\therefore\ S=18000.$$

2. Find the sum of the series $1+3+5+7+9+$, &c., continued to 101 terms.

Ans. 10201.

3. Find the sum of the series 2, 9, 16, 23, &c., to 100 terms.

Ans. 34850.

4. Find the number and the sum of the terms of the series of which 6 is the first term, 796 the last term, and 10 the common difference.

Ans. 80=number of terms,
32080=the sum.

5. Find the common difference and number of terms of a series of which 2 is the first term, 345 the last term, and 8675 the sum of the terms.

Ans. 50=number of terms,
7=common difference.

6. A body falls in a vacuum $16\frac{1}{12}$ feet during the first second, and in each succeeding second $32\frac{1}{6}$ feet more than in the one immediately preceding. If a body fall during the space of 20 seconds, how many feet will it fall in the last second, and how many in the whole time?

Ans. $627\frac{1}{4}$ feet during the last second, and
$6433\frac{1}{3}$ feet in the whole term.

7. Find d when a, l, and n are known.

$$Ans.\ d=\frac{l-a}{n-1}.$$

8. Find d and n when a, l, and S are known.

$$Ans.\ n=\frac{2S}{a+l},$$

$$d=\frac{(l+a)(l-a)}{2S-(a+l)}.$$

9. Find l and n when a, d, and S are known.

$$Ans.\ n=\frac{d-2a\pm\sqrt{(d-2a)^2+8dS}}{2d},$$

$$l=a+(n-1)d.$$

10. Find the last term and number of terms of a series of which the first term is 3, the common difference is 4, and the sum of the terms 105.

Ans. 7=number of terms,
27=last term.

11. Find the sum of the odd numbers from 1 to 99.

Ans. 2500.

12. Find the sum of the even numbers from 2 to 100. *Ans.* 2550.

13. A person bought 47 sheep, and gave 1 shilling for the first, 3 for the second, 5 for the third, and so on. What did all the sheep cost?

Ans. £110 9*s.*

(129.) *To find a number of arithmetical means between two given numbers.*

If m be the number of means, and a and b the given numbers, it is evident that the number of terms of this progression will be $m+2$. Find the common difference; thus, $l=a+(n-1)d$ or $d=\frac{l-a}{n-1}$, becomes

$$d=\frac{b-a}{m+2-1}=\frac{b-a}{m+1}.$$

SECTION II.

GEOMETRICAL PROGRESSION.

(130.) *A Geometrical Progression*, or, *a progression by quotients*, is a series of terms which increase or decrease by the multiplication of a constant quantity.

This constant multiplier is called the *ratio* of the progression.

If the ratio is a positive whole number, the progression is an *increasing* one.

If the ratio is a proper fraction, the progression is a *decreasing* one.

Let a=the first term,

l =the last term,

r = the common ratio,
n = the number of terms,
S = the sum of the series,

we shall have, for the successive terms to n terms,

$$a,\ ar,\ ar^2,\ ar^3\ \ldots\ ar^{n-1}.$$

Then $S=a+ar+ar^2+ar^3+\ ..\ ar^{n-1}\ \ .\ \ .\ \ \ (A)$

Multiply both members of this equation by r, we have

$$Sr=ar+ar^2+ar^3+\ \ldots\ ar^{n-1}+ar^n\ \ .\ \ \ (B)$$

Subtracting (A) from (B),

$$S(r-1)=ar^n-a;$$

$$\therefore\ S=\frac{a(r^n-1)}{r-1}\ \ .\ \ .\ \ .\ \ .\ \ \ (1)$$

If the series be a decreasing one, or r a fraction, it will be more convenient to change the signs of the numerator and denominator in the above; then

$$S=\frac{a(1-r^n)}{1-r}\ \ .\ \ .\ \ .\ \ .\ \ .\ \ \ (2)$$

To find the sum of an increasing geometrical progression, we have by (1) the following

RULE.

Raise the ratio to a power whose exponent denotes the number of terms, subtract unity, multiply the remainder by the first term, and divide by the ratio minus one.

Or, if we substitute the last term, $l=ar^{n-1}$, in (1), we shall have

$$S=\frac{rl-a}{r-1}\ \ .\ \ .\ \ .\ \ .\ \ .\ \ \ (3)$$

That is, to find the sum of any number of terms of a geometrical progression,

Multiply the last term by the ratio, subtract the first term from this product, and divide the remainder by the ratio less one.

(131.) The two equations,

$$l=ar^{n-1}$$

$$S=\frac{a(r^n-1)}{r-1},$$

furnish the means of determining either two of the quantities a, l, r, n, and S, when the other three are given.

But since n is an exponent, *its* value can be found only by the solution of an exponential equation.

(132.) If the series is decreasing, and we suppose it to be continued to an infinite number of terms, the last term may be taken equal to zero. In which case equation (3) becomes, by this substitution, and changing the signs of the numerator and denominator (*Art.* 67),

$$S=\frac{a}{1-r}.$$

Hence, to find the sum of an infinite number of terms of a decreasing geometrical progression, we have the following

RULE.

Divide the first term by unity, diminished by the ratio.

(133.) If we wish to find any number of geometrical means between two numbers, it will first be necessary to obtain the ratio.

If m be the number of means, and a and b the given numbers, it is evident that the number of terms will be $m+2$.

We have $l=ar^{n-1}$, or $r=\sqrt[n-1]{\frac{l}{a}}$.

Substituting b for l and $m+2$ for n, we have

$$r=\sqrt[m+1]{\frac{b}{a}};$$

that is, *divide one of the given numbers by the other, then extract that root of the quotient whose index is one more than the required number of means, and this will be the ratio.*

EXAMPLES.

1. Required the sum of the series 1, 2, 4, 8, 16, &c., to 10 terms. *Ans.* 1023.

2. Required the sum of the first 8 terms of the series 2, 6, 18, 54, &c. *Ans.* 6560.

3. Given the first term 64, the ratio $\frac{1}{4}$, and the number of terms 12, to find the last term.

Ans. $\frac{1}{65536}$.

4. Given the first term 160, the last term 38880, and the number of terms 6, to find the ratio.

Ans. 3.

5. Find the sum of the *infinite* progression of which the first term is 1, and the ratio $\frac{1}{2}$. *Ans.* 2.

6. Find the sum of the infinite progression of which the first term is $\frac{7}{10}$, and the ratio $\frac{1}{10}$.

Ans. $\frac{7}{9}$.

7. Find r in an infinite progression when a and S are known.

Ans. $r=1-\frac{a}{S}$.

8. Find the ratio of an infinite progression of which the first term is 17, and the sum 18.

Ans. $\frac{1}{18}$.

THEORY OF PERMUTATIONS AND COMBINATIONS.

(134.) The *permutations* of any number of quantities are the changes which these quantities may take with respect to their order.

Thus, the three quantities a, b, c, may be permuted in this manner, *abc*, *acb*, *bac*, *bca*, *cab*, *cba*; or, if we permute them *two* and *two*, we shall have *ab*, *ac*, *ba*, *bc*, *ca*, *cb*.

(135.) *To find the number of permutations of* n *quantities taken* m *and* m *together.*

Let a, b, c, d, ... k, be the n quantities.

The number of permutations of these n quantities, taken singly, is evidently n.

Again, if we remove a, there will be $(n-1)$ quantities left; that is,

$$b, c, d, \ldots\ldots k.$$

Writing a before each of these $(n-1)$ quantities, we shall have

$$ab, ac, ad, \ldots\ldots ak;$$

that is, $n-1$ permutations of the n quantities, taken two and two, in which a stands first. Reasoning in the same manner for b, we shall have $n-1$ permutations of the n quantities, taken two and two, in which

b stands first, and in like manner for each of the n quantities in succession; therefore the whole number of permutations will be

$$n(n-1).$$

The number of permutations of n quantities, taken three and three together, is $n(n-1)(n-2)$. For, since there are n quantities, if we remove a there will remain $n-1$ quantities; but, by what precedes, writing $(n-1)$ for n, the number of permutations of $n-1$ quantities, taken two and two, is $(n-1)(n-2)$; now, writing a before each of these $(n-1)(n-2)$ permutations, we shall have $(n-1)(n-2)$ permutations of the n quantities, taken three and three, in which a stands first. Operating in the same manner for b, we shall have $(n-1)(n-2)$ permutations of the n quantities, taken three and three, in which b stands first, and so on for each of the n quantities in succession; hence the whole number of permutations will be

$$n(n-1)(n-2).$$

In a manner entirely similar we can prove that the number of permutations of n quantities, taken four and four, will be

$$n(n-1)(n-2)(n-3).$$

We readily perceive that a certain relation exists between the numerical part of the expressions and the class of permutations to which they correspond.

For example, the number of permutations of n quantities, taken *two* and *two*, is

$$n(n-1) \text{ or } n(n-2+1).$$

Taken *three* and *three*, it is

$$n(n-1)(n-2) \text{ or } n(n-1)(n-3+1).$$

Taken *four* and *four*, it is

$$n(n-1)(n-2)(n-3) \text{ or } n(n-1)(n-2)(n-4+1).$$

Hence we may conclude that the number of permutations of n quantities, taken m and m together, will be

$$n(n-1)(n-2)(n-3)\ldots\ldots(n-m+1) \quad . \quad (1)$$

(136.) If we suppose that each permutation comprehends all the n letters; or, in other words, if $m=n$, formula (1) becomes

$$n(n-1)(n-2)(n-3)\ldots\ldots 1.$$

or $$1, 2, 3, \ldots\ldots (n-1)n \quad . \quad . \quad . \quad (2)$$

the number of permutations of n quantities taken all together.

EXAMPLES.

1. Required the number of permutations of 8 letters, taken 5 and 5 together.

Ans. 6720.

2. Required the number of permutations of 8 letters, taken all together.

Ans. 40320.

(137.) The *combinations* of any number of quantities signify the different collections which may be formed of three quantities, without regard to the *order* in which they are arranged in each collection. Each combination must, therefore, have one letter different from any other of the combinations.

Thus the quantities a, b, c, when taken *all together*, will form six different permutations, but only one combination; and taken *two* and *two*, they will form six permutations, and but three combinations, ab, ac, bc.

(138.) *To find the number of combinations of* n *quantities, taken* m *and* m *together.*

Each of these m quantities being separately permuted, will furnish $1, 2, 3, \ldots . m$ permutations, which, being multiplied by the whole number of combinations, will give the whole number of permutations of n quantities, taken m and m. Therefore,

The whole number of permutations, divided by the number of permutations of each combination, will give the number of combinations of n *quantities, taken* m *and* m.

$$\therefore \frac{n(n-1)(n-2)\ldots\ldots(n-m+1)}{1\,.\,2\,.\,3 \quad \ldots\ldots \quad (m)} \quad . \quad (3)$$

is the formula required.

EXAMPLES.

1. Required the number of combinations of 9 letters, taken 3 and 3. *Ans.* 84.

2. Required the number of combinations of 9 letters, taken 4 and 4. *Ans.* 126.

BINOMIAL THEOREM.

(139.) If we multiply

	$x+a$
by	$x+b$
we shall have	$x^2+(a+b)x+ab$
Again, multiply by	$x+c$
we shall have	$x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$
Again, multiply by	$x+d$
we have	$x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$ $+(abc+abd+acd+bcd)x+abcd.$

We perceive in each of these products that *the coefficient of* x *in the first term is one; the coefficient of the second term is the sum of the second terms of*

the binomial factors; the coefficient of the third term is the sum of all their combinations, taken two and two; the coefficient of the fourth term is the sum of all their combinations, taken three and three, &c.

If now we make a, b, c, d, &c., all equal to each other, the product $(x+a)(x+b)(x+c)\ldots\ldots$ becomes $(x+a)^n$.

The coefficient of the second term becomes na; the coefficient of the third term becomes a^2, repeated as many times as there are different combinations of n letters, taken two and two, or

$$\frac{n(n-1)}{1.2}a^2.$$

The coefficient of the fourth term becomes a^3, repeated as many times as there are different combinations of n letters, taken three and three, or

$$\frac{n(n-1)(n-2)}{1.2.3}a^3, \text{ \&c.}$$

If we put the development of any power of a binomial under the form

$$(1+x)^n = A_0 + A_1x + A_2x^2 + \ldots\ldots A_mx^m + \ldots,$$

so that A_m indicates the coefficient of the mth power of x, as A_2 is the coefficient of x^2, then it will be found that

$$A_0=1,\ A_1=\frac{n}{1},\ A_2=\frac{n(n-1)}{1.2},\ A_3=\frac{n(n-1)(n-2)}{1.2.3}.$$

Now, in order to deduce the expression for A_m, or the coefficient of *any* power in the development, we remark that the value of A_1 is a fraction having one factor in each member, that A_2 is a fraction having two factors in each member, that A_3 is a fraction having

three factors in each member, &c.; hence we conclude that A_m is a fraction having m factors in each member. We next remark that the factors in the numerator successively decrease in value by a unit, the first factor being n, and therefore the last in the expression for A_m must be $n-(m-1)$ or $n-m+1$; and the factors in the denominator successively increase in value by a unit, the first factor being 1, and therefore the last in the expression for A_m must be $1+(m-1)$ or m; hence

$$A_m=\frac{n(n-1)(n-2)\ldots\ldots(n-m+1)}{1\,.\,2\,.\,3\quad\ldots\ldots\quad m}.$$

The numerator expressing the continued product of all numbers differing by unity from n to $n-m+1$ inclusive, and the denominator the continued product of these from 1 to m inclusive; and if in this result we write, instead of m, the successive values 1, 2, 3, 4, &c., we shall get the particular values for A_1, A_2, A_3, &c.

We can express the law of continuity among such a series of numbers by an algebraic relation between any two or more consecutive ones in the series. Thus,

$$\frac{A_1}{A_0}=\frac{n}{1},\ \frac{A_2}{A_1}=\frac{n-1}{2},\ \frac{A_3}{A_2}=\frac{n-2}{3},\ \&c.;$$

hence we may conclude that

$$\frac{A_m}{A_{m-1}}=\frac{n-m+1}{m};$$

or, by clearing this first member of its denominator,

$$A_m=\frac{n-m+1}{m}A_{m-1}=A_{m-1}\,.\,\frac{n-m+1}{m},$$

which, together with the fact that $A_0=1$, makes known the whole of the coefficients. Thus, since $A_0=1$,

$$A_1 = A_0 \cdot \frac{n-1+1}{1} = \frac{n}{1},$$

$$A_2 = A_1 \cdot \frac{n-2+1}{2} = \frac{n(n-1)}{1 \cdot 2},$$

$$A_3 = A_2 \cdot \frac{n-3+1}{3} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3},$$

&c., &c., &c., &c.

If we write the above thus,

$$A_0 = 1,$$

$$A_1 = \frac{n}{1} A_0,$$

$$A_2 = \frac{n-1}{2} A_1,$$

$$A_3 = \frac{n-2}{3} A_2,$$

$$\vdots \qquad \vdots$$

$$A_m = \frac{n-m+1}{m} A_{m-1}.$$

The continued product of the two members of these equation gives

$$A_0 A_1 A_2 A_3 \ldots A_{m-1} A_m = \frac{n(n-1)(n-2)\ldots(n-m+1)}{1 \cdot 2 \cdot 3 \quad \ldots \quad m} A_0 A_1 A_2 \ldots A_{m-1},$$

and, dividing both members by $A_0 A_1 A_2 \ldots A_{m-1}$, we have

$$A_m = \frac{n(n-1)(n-2)\ldots(n-m+1)}{1 \cdot 2 \cdot 3 \quad \ldots \quad m}.$$

(140.) Many remarkable properties of these quantities may be deduced from these relations among them. Thus,

1°. If we put $S.A_n$ to represent the sum of the coefficients, then, making $x=1$, we shall have

$$(1+1)^n=A_0+A_1+A_2+\ldots A_n,$$

or

$$2^n=S.A_n.$$

That is, *the sum of the coefficients of all the terms of the formula for the binomial is equal to the* n*th power of* 2.

2°. If n be an odd whole number, or $n=2n'+1$, by making $m=n'+1$, we shall have

$$A_{n'+1}=\frac{2n'+1-n'-1+1}{n'+1}A_{n'}=A_{n'}.$$

That is, *the coefficients of the two middle terms of the development are equal to each other.*

3°. If n be any integer, the coefficient standing mth from the last, or A_n, may be represented by A_{n-m}, and writing $n-m$, instead of m, in the expression for A_m, we have

$$A_{n-m}=\frac{n(n-1)(n-2)\ldots(n+1)}{1.2.3\quad\ldots n-m}$$

$$=\frac{n(n-1)(n-2)\ldots(n-m+1)(n-m)(n-m-1)\ldots(m+1)}{1.2.3\quad\ldots\quad m\quad.m+1\quad m+2\quad\ldots n-m}$$

$$=\frac{n(n-1)(n-2)\ldots(n-m+1)}{1.2.3\quad\ldots\quad m}$$

$$=A_m.$$

Or, *the coefficients of the terms equally distant from the two extremes are equal to each other.*

(141.) From the above we perceive that the nth power of $(x+a)$ is

$$(x+a)^n=x^n+nx^{n-1}a+\frac{n(n-1)}{1.2}x^{n-2}a^2+\frac{n(n-1)(n-2)}{1.2.3}$$

$$x^{n-3}a^3+\ldots+nxa^{n-1}+a^n.$$

(142.) Therefore, in every development of a binomial, *the index of the first quantity begins with that of the given power, and decreases continually by unity in every term to the last, and the indices of the second quantity are* 0, 1, 2, 3, &c.

The coefficient of the first term is unity, that of the second is the index of the power of the first, and for the others, if the coefficient of any term be multiplied by the index of the leading quantity in it, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following.

When n is a positive whole number, the series will consist of $n+1$ terms. In all other cases it will be *infinite.* If we wish the expansion of $(x+a)^{\frac{n}{r}}$, we shall have, by substituting $\frac{n}{r}$ for m in the above series,

$$(x+a)^{\frac{n}{r}}=x^{\frac{n}{r}}+\frac{n}{r}x^{\frac{n}{r}-1}a+\frac{\frac{n}{r}\left(\frac{n}{r}-1\right)}{1\,.\,2}x^{\frac{n}{r}-2}a^2+, \&c.$$

$$=x^{\frac{n}{r}}+\frac{nx^{\frac{n}{r}}}{r}\cdot\frac{a}{x}+\frac{n(n-r)x^{\frac{n}{r}}}{1\,.\,2\,.\,r^2}\cdot\frac{a^2}{x^2}+, \&c.$$

$$=x^{\frac{n}{r}}\{1+\frac{n}{r}\cdot\frac{a}{x}+\frac{n(n-r)}{1\,.\,2r^2}\cdot\frac{a^2}{x^2}+, \&c.\ldots\}$$

which is a general expression for finding the *approximate* value of any binomial, $\frac{n}{r}$ being either *positive* or *negative.*

Examples.

1. Find the value of $(a^2-x^2)^{\frac{3}{4}}$ in a series.

Here, by *Article* 142, the terms without the coefficients are

$$(a^2)^{\frac{3}{4}},\ (a^2)^{-\frac{1}{4}}x^2,\ (a^2)^{-\frac{5}{4}}x^4,\ (a^2)^{-\frac{9}{4}}x^6,\ \&c.$$

The coefficient of the 1st term is 1,

" " " 2d " $\frac{3}{4}$,

" " " 3d " $\frac{\frac{3}{4}(-\frac{1}{4})}{1\,.\,2}=-\frac{3}{32}$,

" " " 4th " $\frac{\frac{3}{4}(-\frac{1}{4})(-\frac{5}{4})}{1\,.\,2\,.\,3}=\frac{5}{128}$;

$$\therefore\ (a^2-x^2)^{\frac{3}{4}}=a^{\frac{3}{2}}-\frac{3}{4}\cdot\frac{x^2}{a^{\frac{1}{2}}}-\frac{3}{32}\cdot\frac{x^4}{a^{\frac{5}{2}}}-\frac{5}{128}\cdot\frac{x^6}{a^{\frac{9}{2}}}-,\ \text{\&c.}$$

(*Art.* 43).

Or, if we multiply both numerator and denominator of every term except the first by $a^{\frac{3}{2}}$, we shall have

$$(a^2-x^2)^{\frac{3}{4}}=a^{\frac{3}{2}}-\frac{3}{4}\cdot\frac{a^{\frac{3}{2}}x^2}{a^2}-\frac{3}{32}\cdot\frac{a^{\frac{3}{2}}x^4}{a^4}-\frac{5}{128}\cdot\frac{a^{\frac{3}{2}}x^6}{a^6}-,\ \text{\&c.}$$

$$=a^{\frac{3}{2}}\{1-\frac{3x^2}{2^2a^2}-\frac{3x^4}{2^5a^4}-\frac{5x^6}{2^7a^6}-,\ \text{\&c.}\}.$$

2. Convert $(1-x^3)^{\frac{1}{3}}$ into an infinite series.

Ans. $1-\frac{x^3}{3}-\frac{x^6}{3^2}-\frac{5x^9}{3^4}-\frac{10x^{12}}{3^5}$.

3. Convert $(a^2+x^2)^{\frac{1}{2}}$ into an infinite series.

Ans. $a+\frac{x^2}{2a}-\frac{x^4}{2.4a^3}+\frac{3x^6}{2.4.6a^5}-\frac{3.5x^8}{2.4.6.8a^7}+$, &c.

4. Convert $\frac{1}{(a+b)^2}$ or its equal $(a+b)^{-2}$ into an infinite series. *Ans.* $\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-$, &c.

5. Convert $\frac{a^2}{a+x}$ into an infinite series.

Ans. $a-x+\frac{x^2}{a}-\frac{x^3}{a^2}+\frac{x^4}{a^3}-$, &c.

6. Convert $\left(1-\frac{x}{2}\right)^{\frac{1}{3}}$ into an infinite series.

$$Ans.\ 1-\frac{x}{6}-\frac{x^2}{36}-\frac{5x^3}{648}-, \ \&c.$$

7. Convert $(1-x)^{-1}$ into an infinite series.

$$Ans.\ 1+x+x^2+x^3+x^4+, \ \&c.$$

8. Convert $(a+x)^{-4}$ into an infinite series.

$$Ans.\ \frac{1}{a^4}-\frac{4x}{a^5}+\frac{10x^2}{a^6}-\frac{20x^3}{a^7}+, \ \&c.$$

9. Convert $(8+2)^{\frac{1}{3}}$ into an infinite series.

$$Ans.\ 2\{1+\frac{1}{3.4}-\frac{2}{3.6.4^2}+\frac{2.5}{3.6.9.4^3}, \ \&c.\}.$$

10. Convert $(1+x)^{\frac{1}{2}}$ into an infinite series.

$$Ans.\ 1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}+, \ \&c.$$

11. Convert $(a+x)^{\frac{1}{3}}$ into an infinite series.

$$Ans.\ a^{\frac{1}{3}}\{1+\frac{x}{3a}-\frac{2x^2}{3.6a^2}+\frac{2.5x^3}{3.6.9a^3}-, \ \&c.\}.$$

12. Convert $\sqrt{2}$ or $(1+1)^{\frac{1}{2}}$ into an infinite series.

$$Ans.\ 1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}-, \ \&c.$$

INDETERMINATE COEFFICIENTS.

([redacted].) We have seen (*Art.* 141) that any binomial of the form $(a+b)^n$ may be expanded in a series of the form

$$a^n+na^{n-1}b+\frac{n(n-1)}{1\ .\ 2}a^{n-2}b^2+, \ \&c.,$$

whatever may be the value of n; but, although this

theorem is very extensive in its application, mathematicians have invented another method for the development of quantities, more simple in its first principles and more general in its application. This is called the *method of indeterminate coefficients.* We shall proceed to explain its nature.

In order to convey an idea of this method, let it be required to develop the expression

$$\frac{a}{a'+b'x},$$

in a series ascending by powers of x. It is manifest that such a development is possible, for $\frac{a}{a'+b'x}$ may be put under the form $a(a'+b'x)^{-1}$, and, by applying the Binomial Theorem to this expression, we shall obtain a series ascending regularly by powers of x. Let us assume

$$\frac{a}{a'+b'x}=\text{A}+\text{B}x+\text{C}x^2+\text{D}x^3+\text{E}x^4+\text{F}x^5+\ldots, \text{ \&c.} \quad (1)$$

where A, B, C, D, E, F, &c., are quantities involving a, a', b', but independent of x. Coefficients whose value we are required to determine, and which, for that reason, are called *indeterminate coefficients*, or, more properly, *coefficients to be determined.*

In order to ascertain the value of these coefficients, let us multiply the two members of equation (1) by $(a'+b'x)$; then, arranging the result according to the powers of x, and transposing a, we find

$$0=\text{A}a'-a+(\text{B}a'+\text{A}b')x+(\text{C}a'+\text{B}b')x^2+(\text{D}a'+\text{C}b')x^3+, \text{ \&c.} \quad \ldots \quad (2)$$

We may here remark, that if we suppose the values of A, B, C, D, properly determined, equation (1) must hold good, whatever may be the value of x, and in like manner must equation (2) also.

Suppose that $x=0$, this last equation becomes

$$Aa'-a=0,$$

from which $$A=\frac{a}{a'}.$$

A being equal to $\frac{a}{a'}$, when $x=0$, must preserve the same value, whatever may be the value assigned to x; for, by hypothesis, A is independent of x, therefore, whatever may be the value of x, equation (2) reduces itself to

$$0=(Ba'+Ab')x+(Ca'+Bb')x^2+(Da'+Cb')x^3+\ldots,$$

or, dividing by x,

$$0=(Ba'+Ab')+(Ca'+Bb')x+(Da'+Cb')x^2+\ldots \quad (3)$$

Since this equation must hold good, whatever may be the value assigned to x, let us make $x=0$, then

$$Ba'+Ab'=0;$$

$$\therefore\ B=-\frac{Ab'}{a'}$$

$$=-\frac{b'}{a'}\times\frac{a}{a'}$$

$$=-\frac{ab'}{a'^2}$$

Since B must preserve its value whatever may be the value of x, we may suppress in (3) the first term $Ba'+Ab'$, which disappears by the value of B, and, dividing both members of the equation by x, it becomes

$$0=Ca'+Bb'+(Da'+Cb')x+(Ea'+Db')x^2+\ldots$$

Let us, again, make $x=0$, we shall have

$$Ca'+Bb'=0;$$

$$\therefore\ C=-\frac{Bb'}{a'}$$

$$=-\frac{b'}{a'}\times-\frac{ab'}{a'^2}$$

$$=\frac{ab'^2}{a'^3}.$$

In like manner, we shall find

$$D=-\frac{ab'^3}{a'^4},$$

and so for all the rest. We at once perceive that each coefficient is found by multiplying the one preceding by $-\frac{b'}{a'}$; hence the series will be

$$\frac{a}{a'+b'x}=\frac{a}{a'}-\frac{ab'}{a'^2}x+\frac{ab'^2}{a'^3}x^2-\frac{ab'^3}{a'^4}x^3+\frac{ab'^4}{a'^5}x^4-, \text{ \&c.}$$

If we reflect upon the reasoning employed in the process we have just executed, we shall at once perceive that the fundamental principle of the method of indeterminate coefficients consists in this, that

If an equation of the form

$$0=A+Bx+Cx^2+Dx^3+, \text{ \&c.},$$

(A, B, C, D, &c., *being coefficients independent of* x) *hold good, whatever be the value of* x, *then each of the coefficients must separately be equal to zero.*

In fact, since the coefficients are independent of x, if we can determine their value by making particular suppositions with regard to the value of x, these values must still be the same, whatever value we may afterward assign to x.

But by making $x=0$, we find $A=0$, and both members of the equation being divided by x, it becomes

$$0=B+Cx+Dx^2+, \&c.$$

Making $x=0$ in this new equation, we find

$$B=0,$$

and the original equation is reduced, after dividing by x, to

$$0=C+Dx+, \&c.,$$

and so on; we then have

$$A=0,\ B=0,\ C=0,\ D=0,\ \&c.,$$

separately. And in this manner we obtain as many equations as there are coefficients A, B, C, D, &c., to be determined.

This principle may be enunciated under a different form. *If we have an equation of the form*

$$a+bx+cx^2+dx^3+, \&c.=a'+b'x+c'x^2+d'x^3+, \&c.,$$

and it holds good, whatever be the value of x, *then the coefficients of the terms affected by the same powers of* x *in the two members of the equation are respectively equal to each other.*

For if we transpose all the terms into the first member of the equation, the equation will be of the same form as that given above, from whence we may conclude that

$$a-a'=0,\ b-b'=0,\ c-c'=0,\ d-d'=0;$$

and $\quad\therefore\ a=a',\ b=b',\ c=c',\ d=d'.$

The method above elucidated may be summed up in the following

RULE.

1. *Assume the proposed expression equal to a series with unknown coefficients.*

2. *Clear the equation of fractions, or raise it to its proper power.*

3. *Equate the coefficients of homologous terms, and find the value of each;* or, *transpose all the terms into one member, and equate the coefficients with zero.*

EXAMPLES.

1. Convert $\frac{1-x}{1-2x-3x^2}$ into a series by the method of indeterminate coefficients.

Assume $\frac{1-x}{1-2x-3x^2}=A+Bx+Cx^2+Dx^3+$, &c.

Clearing of fractions and transposing, we have

$(A-1)+(B-2A+1)x+(C-2B-3A)x^2+(D-2C-3B)x^3+$, &c.$=0$.

Hence we must have

$$\begin{aligned}&A-1=0, \text{ or } A=1,\\ &B-2A+1=0, \text{ or } B=1,\\ &C-2B-3A=0, \text{ or } C=5,\\ &D-2C-3B=0, \text{ or } D=13,\\ &\quad\quad \&c., \quad\quad\quad \&c.;\end{aligned}$$

$$\therefore \frac{1-x}{1-2x-3x^2}=1+x+5x^2+13x^3+, \&c.$$

2. Convert $\frac{1-x}{1+x}$ into an infinite series.

Ans. $1-2x+2x^2-2x^3+2x^4-$, &c.

3. Convert $(a^2-x^2)^{\frac{1}{2}}$ into an infinite series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-$, &c.

4. Convert $\frac{1+2x}{1-3x}$ into an infinite series.

Ans. $1+5x+15x^2+45x^3+$, &c.

5. Convert $(1+x)^{\frac{1}{2}}$ into an infinite series.

Ans. $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+$, &c.

(144.) By the method of indeterminate coefficients we can develop $\frac{x^m-y^m}{x-y}$; for if we adopt the same notation as *Art.* 139, we shall, if we assume

$$\frac{x^m-y^m}{x-y}=A_0+A_1y+A_2y^2+A_3y^3+\ldots A_my^m+, \text{ \&c.} \quad (1)$$

Clearing this equation of fractions and transposing, we have

$$0=A_0x-x^m+(A_1x-A_0)y+(A_2x-A_1)y^2+\ldots$$
$$(A_mx-A_{m-1})y^m+y^m\ldots+, \text{ \&c.} \quad (2)$$

Hence we must have

$$A_0x-x^m=0 \;\therefore\; A_0x=x^m \;\therefore\; A_0=x^{m-1} \quad (3)$$
$$A_1x-A_0=0 \;\therefore\; A_1x=A_0 \;\therefore\; A_1=x^{m-2} \quad (4)$$
$$A_2x-A_1=0 \;\therefore\; A_2x=A_1 \;\therefore\; A_2=x^{m-3} \quad (5)$$

and $A_nx-A_{n-1}=0 \;\therefore\; A_nx=A_{n-1} \;\therefore\; A_n=x^{m-n-1}$. (6)

Equating the coefficients of y^m to zero, we have

$$A_mx-A_{m-1}+1=0, \text{ or } A_mx-A_{m-1}=-1 \quad (7)$$

If in (6) we put $m-1$ for n, we shall have

$$A_{m-1}=x^0=1.$$

Substitute this value of A_{m-1} in (7), we obtain

$$A_mx-1=-1,$$

or

$$A_mx=0;$$
$$\therefore A_m=0.$$

And hence all the succeeding values of the coefficients will become zero, and the division will terminate with that term whose coefficient is A_{m-1}, or where the exponent of y is $(m-1)$.

These values being substituted in (1), give

$$\frac{x^m-y^m}{x-y}=x^{m-1}+x^{m-2}y+x^{m-3}y^2+\ldots xy^{m-2}+y^{m-1}.$$

SUMMATION OF INFINITE SERIES.

(145.) Though an infinite series consists of an unlimited number of terms, yet in many cases it is not difficult to find the sum of the terms; for, generally, it is a progression of quantities decreasing according to some regular law.

(146.) Let $\frac{q}{r}$ and $\frac{q}{r+p}$ be two fractions; then

$$\frac{q}{r}-\frac{q}{r+p}=\frac{pq}{r(r+p)};$$

or, by dividing by p, we have

$$\frac{1}{p}\left\{\frac{q}{r}-\frac{q}{r+p}\right\}=\frac{q}{r(r+p)}.$$

We see, therefore, that any fraction of the form $\frac{q}{r(r+p)}$ is equal to the $\frac{1}{p}$th part of the difference between the two fractions $\frac{q}{r}$ and $\frac{q}{r+p}$; hence, if there be any series of fractions of the form $\frac{q}{r(r+p)}$, we may find the sum of the series by taking the pth part of the difference of the above fractions, when this difference is known.

EXAMPLES.

1. Required the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$, &c., continued to infinity.

Here $q=1$, $p=1$; also, $r=1, 2, 3, 4, 5$, &c., successively;

$$\therefore \frac{1}{p}\left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+, \text{ \&c., } \textit{ad infin.} \\ -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+, \text{ \&c.}\right) \textit{ ad infin.} \end{array}\right\}=1=\text{sum.}$$

2. Required the sum of the above series to n terms.

We perceive that the denominator of the second term of the fraction $\frac{q}{r}$ is 2, the third 3, &c.; $\therefore$ the denominator of the nth term must be n. We also see that the denominator of the *first* term of the fraction $\frac{q}{r+p}$ is 2, that of the second 3, &c.; $\therefore$ the denominator of the nth term must be $n+1$; hence we have

$$\left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+ \quad \ldots\ldots \frac{1}{n} \\ -\left(\frac{1}{2}+\frac{1}{3}+\right) \ldots\ldots \frac{1}{n}+\frac{1}{n+1} \end{array}\right\}=1-\frac{1}{n+1}=\frac{n}{n+1}.$$

3. Required the sum of the series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+$, &c., continued to infinity. *Ans.* $\frac{1}{2}$.

4. Required the sum of the above series to n terms.

Ans. $\frac{n}{2n+1}$.

5. Required the sum of the series

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+, \text{ \&c.,}$$

Ans. $\frac{11}{18}$.

CHAPTER VII.

DEMONSTRATION OF THE RULE FOR FINDING THE GREATEST COMMON MEASURE—CONTINUED FRACTIONS—LOGARITHMS—EXPONENTIAL EQUATIONS—INTEREST AND ANNUITIES. (See *Art.* 56.)

(147.) Let P and p be any two polynomials whatever, P being the greater; then

$$\begin{array}{r|l} \mathrm{P} & \underline{p} \\ \underline{pq} & q \end{array}$$

$$\mathrm{P}-pq=\mathrm{R} \;\therefore\; \mathrm{P}=pq+\mathrm{R} \quad . \quad . \quad . \quad (1)$$

$$\begin{array}{r|l} p & \underline{\mathrm{R}} \\ \underline{\mathrm{R}q'} & q' \end{array}$$

$$p-\mathrm{R}q'=\mathrm{R}' \;\therefore\; p=q'\mathrm{R}+\mathrm{R}' \quad . \quad . \quad . \quad (2)$$

$$\begin{array}{r|l} \mathrm{R} & \underline{\mathrm{R}'} \\ \underline{q''\mathrm{R}'} & q'' \end{array}$$

$$\mathrm{R}-q''\mathrm{R}'=0 \;\therefore\; \mathrm{R}=q''\mathrm{R}' \quad . \quad . \quad . \quad (3)$$

If we substitute in (1) and (2) the expression $q''\mathrm{R}'$ for its equal R, we shall have

$$\mathrm{P}=pq+q''\mathrm{R}' \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$p=q'q''\mathrm{R}'+\mathrm{R}' \quad . \quad . \quad . \quad . \quad (5)$$

The second member of (5) is divisible by R′; hence the first member, p, or the less polynomial, must also be divisible by the same R′.

Substituting the value of p in equation (4), we obtain

$$\mathrm{P}=qq'q''\mathrm{R}'+q\mathrm{R}'+q''\mathrm{R}' \quad . \quad . \quad . \quad (6)$$

The second member of this last equation can be divided by R′, and therefore the first member can also

be divided by the same quantity; that is, the polynomial P is divisible by R′. Therefore, R′ is a common measure of both polynomials. That it is the *greatest common measure* is evident, because every common measure of P and p is also a measure of $P-pq=R$, and every common measure of p and R is also a measure of $p-q'R=R'$. But the greatest measure of R′ is R′ itself. Hence R′ is the greatest common measure of P and p.

CONTINUED FRACTIONS.

(148.) The name *Continued Fraction* is given to an expression of the form

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+}}}}, \text{ \&c.}$$

That is, a fraction whose numerator is unity, and its denominator an integer increased by a fraction, whose numerator also is unity, &c.

Let us take the above continued fraction, and find its value as it there stands.

Beginning with the last fraction, we have in succession,

$$c+\frac{1}{d}=\frac{cd+1}{d}$$

$$\cfrac{1}{c+\cfrac{1}{d}}=\frac{d}{cd+1}$$

$$b+\cfrac{1}{c+\cfrac{1}{d}}=\frac{b(cd+1)+d}{cd+1}$$

$$\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}=\frac{cd+1}{(bc+1)d+b}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}=\frac{ad(bc+1)+ab+cd+1}{(bc+1)d+b}$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}}=\frac{d(bc+1)+b}{(ab+1)cd+ad+ab+1}=\text{value of the}$$

continued fraction.

(149.) The separate fractions,

$$\frac{1}{a},\ \cfrac{1}{a+\cfrac{1}{b}},\ \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}},$$

are called *approximating* fractions, because each affords a nearer value of the given fraction.

Let us proceed to find the approximating values of the continued fraction,

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+,\ \&c.}}}$$

$$\frac{1}{a}=\text{1st approximate value.}$$

$$\cfrac{1}{a+\cfrac{1}{b}}=\frac{b}{ab+1}=\text{2d approximate value.}$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}=\frac{bc+1}{(ab+1)c+a}=\text{3d approximate value.}$$

Each of these values is easily shown to be more accurate than the preceding; for the second value is what the first becomes by substituting for the denominator a the more accurate denominator $a+\frac{1}{b}$; the third is what the second becomes by substituting for the denominator b the more accurate denominator $b+\frac{1}{c}$, and so on.

(150.) *The numerator of the* nth *approximating fraction is obtained by multiplying the* (n−1)st *numerator by the* nth *denominator contained in the given continued fraction, and adding to the result the numerator of the* (n−2)d *approximating fraction.*

The denominator is obtained in the same way from the two preceding denominators.

(151.) To convert a given irreducible fraction into a continued fraction, we have the following

RULE.

Divide the greater of the two terms of the fraction by the less, and the last divisor continually by the last remainder, as in finding the greatest common measure; then the successive quotients thus found will be the denominators of the several terms of the continued fraction, the numerators of which are always unity.

EXAMPLES.

1. Transform $\frac{351}{965}$ into a continued fraction.

$$\begin{array}{r|l}
965 & 351 \\
702 & \overline{2}=\text{1st quotient,} \\
351 & 263 \\
263 & \overline{1}=\text{2d quotient,} \\
263 & 88 \\
176 & \overline{2}=\text{3d quotient,} \\
88 & 87 \\
87 & \overline{1}=\text{4th quotient,} \\
87 & 1 \\
87 & \overline{87}=\text{5th quotient;} \\
0 &
\end{array}$$

$$\therefore \frac{351}{965}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{87}}}}}$$ the continued fraction required.

2. Transform $\frac{1096}{9119}$ into a continued fraction.

Ans. $$\cfrac{1}{8+\cfrac{1}{3+\cfrac{1}{8+\cfrac{1}{6+\cfrac{1}{7}}}}}.$$

3. Transform $\frac{251}{764}$ into a continued fraction, and find the approximate values.

Ans. App. val. $\frac{1}{3}, \frac{22}{67}, \frac{23}{70}, \frac{114}{347}.$

4. The ratio of the diameter of a circle to the circumference is 1000000 to 3141592. Find approximate values for this ratio.

Ans. $\frac{1}{3}, \frac{7}{22}, \frac{106}{333}$, &c.

5. Transform $\frac{532}{1193}$ into a continued fraction, and find the approximate values.

Ans. $\frac{1}{2}, \frac{4}{9}, \frac{33}{74}.$

6. Transform $\frac{516901}{740785}$ into a continued fraction.

Ans. $$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{8+\cfrac{1}{9}}}}}}}}}.$$

THEORY OF LOGARITHMS.

(152.) *Logarithms* may be considered as the exponents of the powers to which a given or invariable

number, called the base of the system, must be raised in order to produce all the natural numbers. Thus, if

$$a^x=y,\ a^{x'}=y',\ a^{x''}=y'',\ \&c.,$$

then will the exponents x, x', x'', &c., of a be the logarithms of the numbers y, y', y'', &c., in the system whose base is a.

(153.) So that from either of these formulas it appears that *the logarithm of any number is the index of the power to which it is necessary to raise the base of the system in order to produce the number.*

And since the base a in the above expressions can be assumed of any value greater or less than unity, it is plain that there may be an infinite number of systems of logarithms answering to the same natural number.

(154.) It is also further evident, from the first of these equations, that when $y=1$, $x=0$, whatever may be the value of a; and therefore the logarithm of 1 is always 0 in every system of logarithms.

And if $x=1$, it is manifest, from the same equation, that the base a will be equal to y; which base is, therefore, the number whose proper logarithm, in the system to which it belongs, is unity.

(155.) Also, because $a^x=y$ and $a^{x'}=y'$, it follows, from the rule of multiplication, that

$$a^{x+x'}=yy',$$

and therefore, by the definition of logarithms given above,

$$x+x'=\log.\, yy',$$

or

$$\log.\, yy'=\log.\, y+\log.\, y'.$$

And, for a like reason, if any number of equations

$$a^x=y,\ a^{x'}=y',\ a^{x''}=y'',\ \&c.,$$

be multiplied together, we shall have

$$a^{x+x'+x''+,\,\&c.}=yy'y'',\ \&c.;$$

$$\therefore\ x+x'+x''+,\ \&c.=\log.\ yy'y''.$$

From which it is evident *that the logarithm of the product of two or more factors is equal to the sum of the logarithms of those factors.*

(156.) Hence, if all the factors of a given number be supposed equal to each other, and the sum of the logarithm be denoted by m, the preceding property will then become

$$\log.\ y^m=m\log.\ y.$$

From which it appears *that the logarithm of the* m*th power of any number is equal to* m *times the logarithm of that number.*

(157.) In like manner, if the equation $a^x=y$ be divided by $a^{x'}=y'$, we shall have

$$a^{x-x'}=\frac{y}{y'},$$

and, by the definition of logarithms,

$$\log.\ \frac{y}{y'}=\log.\ y-\log.\ y'.$$

Hence *the logarithm of a fraction, or of the quotient arising from dividing one number by another, is equal to the logarithm of the numerator minus the logarithm of the denominator.*

(158.) And if each member of the equation $a^x=y$ be raised to the fractional power denoted by $\frac{m}{n}$, we shall have

$$a^{\frac{m}{n}x}=y^{\frac{m}{n}}.$$

And consequently, taking the logarithms as before,

$$\log. y^{\frac{m}{n}} = \frac{m}{n} \log. y.$$

Hence *the logarithm of a fractional power of any number is found by multiplying the logarithm of the given number by the numerator of the index of that power, and dividing the result by the denominator.*

(159.) And if the numerator m of the fractional index be unity, the above formula will become

$$\log. y^{\frac{1}{n}} = \frac{1}{n} \log. y.$$

Hence *the logarithm of the* n*th root of any number is equal to the* n*th part of the logarithm of that number.*

(160.) Although the properties here mentioned are common to every system of logarithms, it was necessary, for practical purposes, to select some one of them from the rest, and to adapt the logarithms of all the natural numbers to that particular system; and as 10 is the base of our present system of Arithmetic, the same number was chosen for the base of the logarithmic system.

According to this system, we shall have

$$\begin{array}{rcl}
10^0 = 1 & \therefore & \log.\ 1 = 0 \\
10^1 = 10 & \therefore & \log.\ 10 = 1 \\
10^2 = 100 & \therefore & \log.\ 100 = 2 \\
10^3 = 1000 & \therefore & \log.\ 1000 = 3, \\
\&c., & & \&c. \\
10^{-1} = \frac{1}{10} & \therefore & \log.\ .1 = -1 \\
10^{-2} = \frac{1}{100} & \therefore & \log.\ .01 = -2, \\
\&c., & & \&c.,
\end{array}$$

which are evidently a set of numbers in *arithmetical* progression, corresponding to another set in *geometrical* progression.

It is also evident that the logarithm of any number between 1 and 10 will be zero + a decimal; between 10 and 100 it will be 1+ a decimal. The integral part, which is called the index or characteristic, is always *one less* than the number of integers which the natural number consists of, and for decimals it is the number which denotes the distance of the first significant figure from the decimal point.

(161.) *To find a series for the logarithm of any number in terms of the number itself, and the base of the system.*

Let y be any number whose logarithm is x, in a system whose base is a, then

$$a^x=y,$$

or

$$a^{mx}=y^m,$$

or

$$\{1+(a-1)\}^{mx}=\{1+(y-1)\}^m;$$

$\therefore$ by development,

$$1+mx(a-1)+\frac{mx(mx-1)}{1\,.\,2}(a-1)^2+\ldots=1+m(y-1)$$
$$+\frac{m(m-1)}{1\,.\,2}(y-1)^2+\ldots$$

Subtracting unity from each and dividing by m,

$$x(a-1)+\frac{x(mx-1)}{1\,.\,2}(a-1)^2+\ \ldots\ =(y-1)+\frac{m-1}{1\,.\,2}$$
$$(y-1)^2+\ldots$$

Arranging according to powers of m, we have

$$x\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ \ldots\ \}$$
$$+\mathrm{P}m+\mathrm{Q}m^2+\ \ldots$$
$$=\{(y-1)-\tfrac{1}{2}(y-1)^2+\tfrac{1}{3}(y-1)^3-\tfrac{1}{4}(y-1)^4+\ \ldots\ \}$$
$$+\mathrm{P}'m+\mathrm{Q}'m^2+\ \ldots$$

Where P, Q, P$'$, Q$'$ are independent of m, and since m is indeterminate, we have, by comparing coefficients of homologous terms,

$$x(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ldots=(y-1)$$
$$-\tfrac{1}{2}(y-1)^2+\tfrac{1}{3}(y-1)^3-\tfrac{1}{4}(y-1)^4;$$

$\therefore$ x, or the log. $y=$

$$\frac{(y-1)-\tfrac{1}{2}(y-1)^2+\tfrac{1}{3}(y-1)^3-\tfrac{1}{4}(y-1)^4+}{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+}.$$

If we put $\dfrac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+}$, &c.$=$A, we shall have

$$\log.\ y=\text{A}\{(y-1)-\tfrac{1}{2}(y-1)^2+\tfrac{1}{3}(y-1)^3-\tfrac{1}{4}(y-1)^4$$
$$+\tfrac{1}{5}(y-1)^5-,\ \&\text{c.}\},$$

the series required.

The expression for the logarithm of any number is composed of two factors, one dependent on the number, and the other on the base of the system in which the logarithm is taken.

The factor which depends on the base is called the modulus of the system of logarithms.

(162.) If we take the logarithm of y in a new system, whose base is a', and denote it by l', and the modulus by A$'$, we shall have

$$l'.y=\text{A}'\{(y-1)-\tfrac{1}{2}(y-1)^2+\tfrac{1}{3}(y-1)^3-\tfrac{1}{4}(y-1)^4+\ldots\}$$

Hence, calling the first log. of y, l, we shall have

$$l.y : l'y :: \text{A} : \text{A}';\ \text{therefore,}$$

The logarithms of the same number, taken in two different systems, are to each other as the moduli of those systems.

Now, as the modulus of the Naperian system is unity, we shall have, by putting A$'$=1,

$$\text{A}.l'.y=l.y.$$

That is, *the Naperian or hyperbolic logarithm of any number, multiplied by the modulus of another system, is equal to the logarithm of the same number in that system.*

(163.) From the equation

$$A.l'.y=l.y,$$

we have $$l'.y=\frac{l.y}{A},$$

or, *the logarithm of any number divided by the modulus of its system is equal to the Naperian logarithm of the same number.*

(164.) *To find the logarithm of a number in a converging series.*

We have seen that if $a^x=y$, then

$$x=A\left(\frac{y-1}{1}-\frac{(y-1)^2}{2}+\frac{(y-1)^3}{3}-\frac{(y-1)^4}{4}+\right).$$

Let $y=1+y'$ $\therefore$ $y-1=y'$, $x=\log. y=\log. (1+y')$;

$$\therefore \log. (1+y')=A\left(\frac{y'}{1}-\frac{y'^2}{2}+\frac{y'^3}{3}-\frac{y'^4}{4}+\ldots\right).$$

Let $y''=1-y'$ $\therefore$ $y''-1=-y'$; and hence,

$$\therefore \log. (1-y')=A(-\frac{y'}{1}-\frac{y'^2}{2}-\frac{y'^3}{3}-\ldots)$$

$$\therefore \log. (1+y')-\log. (1-y')=2A\left(\frac{y'}{1}+\frac{y'^3}{3}+\frac{y'^5}{5}+\ldots\right)$$

or $$\log. \frac{1+y'}{1-y'}=2A\left(\frac{y'}{1}+\frac{y'^3}{3}+\frac{y'^5}{5}+\right).$$

Now any number y may be put under the form

$$y=\frac{1+\frac{y-1}{y+1}}{1-\frac{y-1}{y+1}}.$$

Substituting, therefore, $\frac{y-1}{y+1}$ for y' in the above series, we have

$$\log. y=2A\left\{\left(\frac{y-1}{y+1}\right)+\frac{1}{3}\left(\frac{y-1}{y+1}\right)^3+\frac{1}{5}\left(\frac{y-1}{y+1}\right)^5+\ldots\right\}.$$

A series which is always convergent, since the denominator of any term is greater than the numerator; and from this series the logarithm of any number may be calculated.

EXPONENTIAL EQUATIONS.

(165.) *Exponential Equations* are such as contain quantities with unknown exponents.

Ex. 1. Given $a^x=b$ to find the value of x.

The logarithms of the two members of this equation give
$$x \log. a=\log. b;$$
$$\therefore\ x=\frac{\log. b}{\log. a},$$
or
$$\log. x=\log. \text{ of } \log b-\log. \text{ of } \log. a.$$

Ex. 2. Given
$$\frac{963x}{1279}=\frac{87}{10}.$$

Clearing the equation of fractions, we have
$$9630x=1279\times 87$$
$$x=\frac{1279\times 87}{9630},$$
or
$$\log. x=\log. 1279+\log. 87-\log. 9630.$$

$$\begin{array}{lr}
\log. 1279= & 3.10687 \\
\log. \quad 87= & \underline{1.93952} \\
 & 5.04639 \\
\log. 9630= & \underline{3.98363} \\
\log. x= \quad . \quad . \quad . \quad . & 1.06276
\end{array}$$

$$\therefore\ x=11.555.$$

Ex. 3. Given $\frac{138}{319}=\frac{153}{2x}$ to find the value of x.

Ans. $x=1768.3$.

Ex. 4. Given $625^x=3125$ to find the value of x.

Ans. $x=1.25$.

Ex. 5. Given $3^x=15$ to find the value of x.

Ans. 2.464.

Ex. 6. Given $10^x=3$ to find the value of x.

Ans. $x=0.477$.

Ex. 7. Given $ab^x+c=\frac{e}{d}$ to find the value of x.

$$Ans.\ x=\frac{\log.\ (de-c)-\log.\ a}{\log.\ b}.$$

COMPOUND INTEREST.

(166.) In the solution of problems in Compound Interest and Annuities, logarithms may be used with great advantage.

Let p represent any principal in dollars,

r the interest of one dollar for 1 year;

then $1+r$ equal the amount of one dollar for 1 year,

and $p(1+r)=$the amount of principal for 1 year;

$\therefore p(1+r)+pr(1+r)=p(1+r)^2=$amount 2d year,

and $p(1+r)^2+pr(1+r)^2=p(1+r)^3=$amount 3d year.

Hence the amount in any number (t) of years will be, if we put A=that amount,

$$p(1+r)^t=A \quad . \quad . \quad . \quad . \quad (1)$$

Examples.

1. What will \$8750 amount to in 12 years at 6 per cent. compound interest?

Here we have $8750(1.06)^{12}=A$;

$$\therefore \text{log. A} = \text{log. } 8750 + 12 \text{ log. } 1.06.$$

$$\begin{aligned} \text{log. } 8750 &= 3.942008 \\ 12 \text{ log. } 1.06 &= \underline{0.303670} \\ \text{log. A} = \quad . \quad . \quad . \quad &= 4.245678 \end{aligned}$$

$$\therefore \text{A} = \$17606.72.$$

2. What will \$5000 amount to in 40 years at 4 per cent. compound interest? *Ans.* 24005.10.

3. The amount of \$400 at compound interest for 9 years was \$569.333; what was the rate per cent.?

$$\text{log. } (1+r) = \frac{1}{t}(\text{log. A} - \text{log. } p),$$

or $$\text{log. } (1+r) = \frac{1}{9}(\text{log. } 569.333 - \text{log. } 400).$$

Ans. .04, or 4 per cent.

ANNNUITIES.

(167.) *To find the amount of an annuity for* t *years at compound interest.*

At the end of the first year the annuity a will become due; at the end of the second year, a second annuity a will become due, together with the interest of the first payment a for the one year, that is, ar; the whole sum upon which interest must now be calculated is $2a+ar$. Thus,

The whole am't at the end of the 1st year $=a$,

" " " " 2d year $=a+a+ar$

$$=a+a(1+r),$$

At the end of the 3d year,

$$=a+a+a(1+r)+ar+ar(1+r)$$
$$=a+a(1+r)+a(1+r)^2,$$

At the end of the 4th year,

$$=a+a(1+r)+a(1+r)^2+a(1+r)^3;$$

$\therefore$ The whole amount at the end of the tth year is

$$a+a(1+r)+a(1+r)^2+a(1+r)^3+\ldots+a(1+r)^{t-1};$$

$$\therefore \mathrm{A}=a\{1+(1+r)+(1+r)^2+(1+r)^3+\ldots(1+r)^{t-1}\}.$$

A geometrical progression,

$$\therefore \mathrm{A}=\frac{a(1+r)^t-1}{r},$$

which in logarithms is

$$\log. \mathrm{A}=\log. a+\log. \{(1+r)^t-1\}-\log. r.$$

Ex. An annuity of \$50 has remained unpaid for 6 years; what sum is now due, compound interest being reckoned at 6 per cent.?

$$\log. \mathrm{A}=\log. 50+\log. \{(1.06)^6-1\}-\log. .06.$$

$$\begin{array}{rr} \log. 50= & 1.698970 \\ \log. (1.06^6-1)= & \underline{\bar{1}.621715} \\ & 1.320685 \\ \log. .06= & \underline{\bar{2}.778151} \\ \$348.61 \quad . \quad . & 2.542534. \end{array}$$

(168.) *To find the present value of an annuity* a *payable for* t *years at compound interest.*

The present value is such a sum as, being put out at interest for t years at the given rate, will amount to the same sum as the annuity.

Let p=the present worth, r=rate per cent.; then

$$p(1+r)^t=\text{amount of the annuity};$$

$$\therefore p(1+r)^t=\frac{a(1+r)^t-1}{r},$$

or

$$p=\frac{a}{r}\cdot\frac{(1+r)^t-1}{(1+r)^t}=\frac{a}{r}\left\{1-\frac{1}{(1+r)^t}\right\}.$$

Hence, to find the present worth of an annuity, *divide the amount of the annuity unpaid by the amount of one dollar for the same number of years.*

Ex. What is the present value of an annuity of \$500, to last for 40 years, compound interest being allowed at the rate of $2\frac{1}{2}$ per cent. per annum?

Ans. \$12551.40.

To find the present value p *of an estate or perpetuity, whose annual rental is* a, *compound interest being reckoned at the rate* r.

The present value of an annuity a to continue for t years is, by the last problem,

$$p=\frac{a}{r}\left\{1-\frac{1}{(1+r)^t}\right\};$$

but if the annuity last forever, then $t=\infty$, and

$$\therefore \frac{1}{(1+r)^t}=\frac{1}{\infty}=0;$$

hence $$p=\frac{a}{r}.$$

That is, *divide the amount of the annual rental by the number representing the rate per cent., the quotient will be the present value.*

Ex. What is the value of an estate whose rental is \$1500, allowing the purchaser 5 per cent. for his money?

Ans. \$30,000, or 20 years' purchase.

CHAPTER VIII.

DERIVATION—GENERAL THEORY OF EQUATIONS.

SECTION I.

ON DERIVATION.

(169.) When quantities are so connected that the value of each is dependent upon that of the others, each is said to be a *function* of the others.

Thus, if $x=ay+b$, x is a function of a, b, and y; but if y is variable while a and b are constants, the more usual method is to regard x simply as a function of y.

In case of a change in the value of a function, arising from an infinitely small change in the value of one of its variables, *the relative rate of change of the function and the variable is called the derivative of the function.*

(170.) *The derivative of the sum of two functions is equal to the sum of their derivatives.*

Let the functions be u and v, and let their values arising from an indefinitely small change h, in the value of the variable, be u' and v', then

$$\frac{u'-u}{h} \text{ and } \frac{v'-v}{h}$$

will be their derivatives; but the increase of their sum will evidently be

$$u'+v'-(u+v),$$

or $$u'-u+v'-v;$$

and therefore the derivative of the sum is

$$\frac{u'-u+v'-v}{h},$$

which is evidently the sum of their derivatives.

By changing the sign of v, it will be shown that *the derivative of the difference of two functions is equal to the difference of their derivatives.*

(171.) *To find the derivative of any power of a variable.*

Let x be the variable and x^n the power, and let x' differ infinitely little from x, the derivative of x^n is then

$$\frac{x'^n-x^n}{x'-x}.$$

But

$$\frac{x'^n-x^n}{x'-x}=x'^{n-1}+x'^{n-2}x+x'^{n-3}b^2+,\ \&c.+\ \ldots\ x'x^{n-2}+x^{n-1}.$$

And when $x'=x$, the value of this quotient is

$$nx^{n-1}$$

for the derivative of x^n.

That is, *the derivative of any power of a variable is found by multiplying by the exponent, and diminishing the exponent by unity.*

(172.) The derivative of mx^n, when m is constant, is nmx^{n-1}.

(173.) *To find the derivative of any power of a function.*

Let the variable be x, the function u, and the power u^n; let x' differ infinitely little from x, and let v be the corresponding value of u. If U is the derivative of u, and U$'$ that of u^n, we have

$$U'=\frac{v^n-u^n}{x'-x} \text{ and } U=\frac{v-u}{x'-x}.$$

But by *Art.* 171, $\dfrac{v^n-u^n}{v-u}=nu^{n-1}$,

which, multiplied by $U=\dfrac{v-u}{x'-x}$, gives

$$U'=\frac{v^n-u^n}{v-u}\cdot\frac{v-u}{x'-x}=\frac{v^n-u^n}{x'-x}=nu^{n-1}U.$$

That is, *the derivative of any power of a function is found by multiplying by the exponent and by the derivative of the function, and diminishing the exponent by unity.*

EXAMPLES.

1. What is the derivative of $x^n+ax^m+bx^n+cx^p$?

Ans. $nx^{n-1}+amx^{m-1}+bnx^{n-1}+cpx^{p-1}$.

2. What is the derivative of x^4?

Ans. $4x^3$.

3. What is the derivative of $(a+bx)^n$?

Ans. $nb(a+bx)^{n-1}$.

(174.) *To find the derivative of the product of two functions.*

Let u and v be the functions, and U and V their derivatives; then, since the derivative is the rate of change of the function to that of the variable, it is manifest that when the variable is increased by the infinitesimal h, that the functions will become

$$u+Uh \text{ and } v+Vh.$$

The product will therefore change from uv to

$$(u+Uh)(v+Vh), \text{ or } uv+vUh+vVh+UVh^2,$$

and the increase of the product is

$$vUh+uVh+UVh^2;$$

the ratio of which to h is

$$vU+uV+UVh,$$

or, neglecting the infinitesimal UVh, it becomes

$$vU+uV.$$

That is, *the derivative of a product of two functions is equal to the sum of the two products obtained by multiplying each function by the derivative of the other function.*

SECTION II.

GENERAL THEORY OF EQUATIONS.

(175.) *Any equation of the* n*th degree* with one unknown quantity, when reduced, may be put under the form

$$Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots M=0 \quad . \quad (1)$$

If we divide this equation by A, and represent the coefficients of the succeeding terms by a, b, c, &c., m, it will become

$$x^n+ax^{n-1}+bx^{n-2}+\ldots m=0 \quad . \quad . \quad (2)$$

(176.) If any root of this last equation be denoted by x', the first member is divisible by $x-x'$. Dividing, we have

$$\begin{array}{r|r|r|l} x^{n-1}+x' & x^{n-2}+\ x'^2 & x^{n-3}+\ x'^3 & x^{n-4}+, \text{\&c.} \\ +a & +ax' & +ax'^2 & \\ & +b & +bx' & \\ & & +c & \end{array}$$

The first term of the preceding quotient is x^{n-1}, and if the coefficients of x^{n-2}, x^{n-3}, &c., be represented by a', b', c', &c., it becomes

$$x^{n-1}+a'x^{n-2}+b'x^{n-3}+, \text{\&c.},$$

and the equation (2) is

$$(x-x')\,(x^{n-1}+a'x^{n-2}+b'x^{n-3}+, \text{\&c.})=0,$$

which is satisfied either by making $x=x'$, or by the roots of the equation

$$x^{n-1}+a'x^{n-2}+b'x^{n-3}+, \&c.=0 \quad . \quad . \quad (3)$$

If we now suppose x'' to be one of the roots of (3), we shall have

$$x^{n-1}+a'x^{n-2}+b'x^{n-3}+, \&c.=(x-x'')\,(x^{n-2}+a''x^{n-3}+, \&c.)=0,$$

which is satisfied by making

$$x=x'';$$

so that x'' is a root of the given equation.

In like manner, we may find the roots x^{III}, x^{IV}, x^{V}, &c., and the given equation may be reduced to the form

$$(x-x')\,(x-x'')\,(x-x'''), \&c.=0,$$

in which the number of factors is the same with the degree of the equation; and hence *the number of roots of an equation is denoted by the degree of the equation.* That is, an equation of the second degree has two roots, one of the third degree has three roots, &c.

All these roots, however, are not necessarily real or rational; they may be irrational, or even imaginary.

EQUAL ROOTS.

(177.) Some of the roots are often equal to each other, and then the number of unequal roots is less than the degree of the equation.

The equation $x^n=a$ would appear to have but one root; that is, $x=\sqrt[n]{a}$,

but the nth root of a must have n different values.

Take the equation $x^3=1$,

or $x^3-1=0$.

Now, as 1 is one value of x, the first member must be divisible by $x-1$; dividing by this factor, we have

$$x^2+x+1=0.$$

Whence, by the rule for equations of the second degree,

$$x=\tfrac{1}{2}(-1+\sqrt{-3}) \text{ and } x=\tfrac{1}{2}(-1-\sqrt{-3}).$$

So we perceive that the three roots are

$$x=1,\ \tfrac{1}{2}(-1+\sqrt{-3},\ \text{and } \tfrac{1}{2}(-1-\sqrt{-3}).$$

(178.) If the roots of the general equation of the third degree, $x^3+ax^2+bx+c=0$, are denoted by x', x'', x''', we shall have

$$x^3+ax^2+bx+c=(x-x')(x-x'')(x-x''')=0,$$

or $x^3+ax^2+bx+c=x^3-(x'+x''+x''')x^2+(x'x''+x'x'''+x''x''')x-x'x''x'''=0.$

Whence

$$a=-(x'+x''+x''')$$
$$b=(x'x''+x'x'''+x''x''')$$
$$c=-x'x''x'''.$$

That is, *the coefficient of* x^2 *is the negative of the sum of the roots, the coefficient of* x *is the sum of the products of the roots multiplied together two and two, and the term which does not contain* x *is the negative of the continued product of the roots.*

(179.) It may be shown in the same manner that, in the equation

$$x^n+ax^{n-1}+bx^{n-2}+cx^{n-3}+, \text{ \&c.}=0,$$

the coefficient of x^{n-1} *is the sum of the roots taken with a negative sign; the coefficient of* x^{n-2} *is the sum of the products of the roots taken two and two; the coefficient of* x^{n-3} *is the negative of the sum of the products of the roots taken three and three; and so on, the last term being the product of the roots taken positively when* n *is an even number, and negatively when* n *is an odd number.*

(180.) *To find the equal roots of an equation.*

Let x' be one of the equal roots which occurs n times as a root of the given equation, the first member of which is, therefore, divisible by $(x-x')^n$. If X denotes the quotient, and Y is the derivative of X, we have

$$(x-x')^n X=0.$$

The derivative of this first member is, *Art.* 174,

$$n(x-x')^{n-1}X+(x-x')^n Y.$$

The factor $(x-x')$ occurs, then, $(n-1)$ times in this derivative of the first member; that is, once less than it occurs in the first member itself. The greatest common divisor of the first member and its derivative must, therefore, consist of the factors $(x-x')$ of the first member, each being repeated once less than in the first member. No one of them is, then, a factor of the common divisor, unless it is more than once a factor of the first member, that is, unless it corresponds to one of the equal roots. Hence we obtain the following

Rule.

Find the greatest common divisor of the first member of the equation and its derivative; equate this common divisor with zero, and the solution of this equation will give the roots required.

Examples.

1. Find all the roots of the equation

$$x^7-9x^5+6x^4+15x^3-12x^2-7x+6=0 \quad . \quad (1)$$

The derivative of this equation is

$$7x^6-45x^4+24x^3+45x^2-24x-7,$$

the greatest common divisor of which and the given equation is

$$x^3-x^2-x+1=0 \quad . \quad . \quad . \quad . \quad (2)$$

We may consider this as a new equation, whose derivative is $$3x^2-2x-1,$$
and the common divisor of this derivative and equation (2) is $x-1$ $\therefore$ $x-1=0$, or $x=1$.
Hence the first member of $x^3-x^2-x+1=0$ must be divisible by $(x-1)^2$

$$\therefore\ x^3-x^2-x+1=(x-1)^2(x+1)=0.$$

The equal roots of the equation are, therefore,

$$x=1 \text{ and } x=-1.$$

And the original equation is divisible by

$$(x-1)^3(x+1)^2,$$

and is found to be

$$(x-1)^3(x+1)^2(x^2+x-6)=0;$$

$$\therefore\ x^2+x-6=0,$$

or $$x^2+x=6.$$

From which we have

$$x=2, \text{ or } x=-3.$$

2. Find all the roots of the equation

$$x^3-15x^2+75x-125=0.$$

Ans. There are three roots, each $=5$.

3. Find all the roots of the equation

$$x^4+12x^3+54x^2+108x+81=0.$$

Ans. Four roots, each $=-3$.

4. Find all the roots of the equation

$$x^3-7x^2+16x-12=0.$$

Ans. $x=2$, 2 and 3.

REAL ROOTS.

(181.) When an equation is reduced to the form

$$Ax^n+Bx^{n-1}+Cx^{n-2}+\ .\ .\ .\ +K=0,$$

and the values of its member, obtained by the substitution of two different numbers for its unknown quan-

tity, are affected by contrary signs, the given equation must have a *real root* comprehended between these two numbers.

For if the value of the less of the two numbers, which are substituted for the unknown quantity, is supposed to be increased by imperceptible degrees until it attains the value of the greater number, the value of the first member must also change by imperceptible degrees, and must pass through all the intermediate values between its two extreme values. But the extreme values are affected with opposite signs, so that zero must be contained between them, and must be one of the values of the first member; that is, there must be a number which, substituted in the first member, reduces it to zero, and this number is consequently a root of the given equation.

(182.) *If the proposed equation has no real root, the value of its first member will always be affected by the same sign, whatever numbers be substituted for the unknown quantity.*

(183.) *When an uneven number of the real roots of an equation are comprehended between two numbers, the values of its first member obtained, by substituting these numbers for* x, *must be affected with opposite signs; but if an even number of roots is contained between them, the values obtained from this substitution must be affected with the same sign.*

For, denote by x', x'', x''', &c., all the roots of the given equation which are contained between the given numbers p and q; the first member of the proposed equation must be divisible by

$$(x-x')(x-x'')(x-x'''), \text{ \&c.}$$

If we denote the quotient of this division by Q, the equation
$$Q=0$$
gives all the remaining roots of the given equation, so that
$$Q=0$$
can not have any real root contained between p and q.

The first member, therefore, being represented by
$$(x-x')(x-x'')(x-x''') \ldots \times Q,$$
becomes
$$(p-x')(p-x'')(p-x''') \ldots \times Q',$$
when we substitute p for x, and denote the corresponding value of Q by Q'; and when we substitute q for x, and denote the corresponding value of Q by Q'', it becomes
$$(q-x')(q-x'')(q-x''') \ldots \times Q''.$$

The quotient of these two results is
$$\frac{(p-x')(p-x'')(p-x''') \ldots Q'}{(q-x')(q-x'')(q-x''') \ldots Q''},$$
which may be written
$$\frac{p-x'}{q-x'} \times \frac{p-x''}{q-x''} \times \frac{p-x'''}{q-x'''} \times \ldots \frac{Q'}{Q''}.$$

Now, since each of the roots x', x'', x''', &c., is included between p and q, the numerator and denominator of each of the fractions
$$\frac{p-x'}{q-x'}, \quad \frac{p-x''}{q-x''}, \quad \frac{p-x'''}{q-x'''}$$
must be affected with opposite signs, and therefore each of these fractions must be negative.

But, since Q' and Q'' must have the same sign (*Art.* 182), the fraction
$$\frac{Q'}{Q''}$$
is positive.

The product of all these fractions is, therefore, *positive* when the number of the fractions is *even;* that is, when *the number of the roots* x', x'', x''', &c., *is even;* and this product is *negative* when the number of these roots is *uneven.* The values which the first member of the given equation obtains by the substitution of p and q for x, must, therefore, be affected with *contrary signs* in the latter case, and with the *same sign* in the former case.

(184.) *Every equation of an uneven degree has at least one real root affected with a sign contrary to that of its last term, and the number of all its roots is uneven.*

Let the equation be

$$x^n+ax^{n-1}+bx^{n-2}+\ .\ .\ .\ m=0,$$

in which n is uneven.

1°. To prove that there is a real root, and that the number of real roots is uneven, every real root must be contained between $+\infty$ and $-\infty$. Now the substitution of $x=+\infty$ gives the value of the first member,

$$\infty^n+a\infty^{n-1}+b\infty^{n-2}+\ .\ .\ .\ m,$$

the first term of which is infinitely greater than any other term, or than the sum of all the other terms. The sign of this result is, therefore, *positive.*

Again, the substitution of $x=-\infty$ gives, since n is uneven,

$$-\infty^n+a\infty^{n-1}+b\infty^{n-2}+\ .\ .\ .\ m,$$

which may be shown, by the above reasoning, to be *negative.*

The given equation must then have at least one real root, and the number of its real roots must be uneven.

2°. To prove that one, at least, of the real roots is affected with a sign contrary to that of the last term.

The substitution of $x=0$ reduces the first member to its last term m.

Comparing this with the above results, we see that if m is *positive*, the given equation must have a real root contained between 0 and $-\infty$, that is, a *negative* root; but if m is *negative*, there must be a real root contained between 0 and $+\infty$, that is, a *positive* root, so that there must always be a root affected with a sign contrary to that of m.

(185.) *The number of real roots, if there are any, of an equation of an even degree must be even, and if the last term is negative, there must be at least two real roots, one positive and the other negative.*

Let the equation be

$$x^n+ax^{n-1}+bx^{n-2}+cx^{n-3}+\ .\ .\ .\ m=0,$$

in which n is even.

1°. To prove that the number of real roots is even.

The substitution of $x=+\infty$ gives, for the first member,

$$\infty^n+a\infty^{n-1}b+\infty^{n-2}+c\infty^{n-3}+\ .\ .\ .\ m,$$

which is *positive.*

The substitution of $x=-\infty$ gives, for the first member,

$$\infty^n-a\infty^{n-1}+b\infty^{n-2}-c\infty^{n-3}+\ .\ .\ .\ m,$$

which is also *positive.*

Hence, if the proposed equation has any real roots, there must be *an even number of them.*

2°. To prove that when m is negative there must be two real roots, the one positive, the other negative.

The substitution of $x=0$ reduces the first member

to its last term m, and this result is, by hypothesis, *negative.*

Comparing this with the above results, we see that there must be a real root between 0 and $+\infty$, and also one between 0 and $-\infty$; that is, the given equation has two real roots, the one positive, and the other negative.

(186.) *The number of real positive roots of an equation is even when its last term is positive, and uneven when the last term is negative.*

The substitution of $x=\infty$ gives, for the first member of the proposed equation, a *positive* result, while the substitution of $x=0$ reduces the first member to its last term m.

Hence, if this last term is positive, the number of real roots contained between 0 and ∞, that is, of positive roots, must be even; and if this last term is negative, the number of these roots must be uneven.

STURM'S THEOREM.

(187.) *Definition.*—A pair of two successive signs, in a row of signs, is called a *permanence* when the two signs are alike, and a *variation* when they are unlike.

(188.) Let the first member of the equation

$$x^n+ax^{n-1}+bx^{n-2}+, \&c.=0$$

be denoted by u, and its derivative by U. Find the greatest common divisor of u and U, and, in performing this process, let the several remainders, after having changed their signs, be represented by

$$U', U'', U''', \&c.$$

Find the row of signs corresponding to the values of

$$u, \mathrm{U}, \mathrm{U}', \mathrm{U}'', \mathrm{U}''', \&c.,$$

for any value p of x, and also for any value q of x.

Then *the difference between the number of permanences of the first row of signs and that of the second is exactly equal to the number of real roots of the proposed equation comprised between* p *and* q.

For the method by which U′, U″, &c., are obtained gives at once, by denoting the successive quotients in the process by k, k', &c.,

$$\begin{aligned} u &= k\mathrm{U} - \mathrm{U}', \\ \mathrm{U} &= k'\mathrm{U}' - \mathrm{U}'', \\ \mathrm{U}' &= k''\mathrm{U}'' - \mathrm{U}''', \\ &\&c., \quad \&c. \end{aligned}$$

1°. Two successive terms of the series can not vanish at the same time, except for a value of x which is one of the equal roots of the equation. For when U″ and U‴, for instance, are zero, the equation

$$\mathrm{U}' = k''\mathrm{U}'' - \mathrm{U}'''$$

gives $$\mathrm{U}' = 0.$$

And in like manner it is shown that

$$\mathrm{U} = 0 \text{ and } u = 0;$$

so that the function and the derivative are both zero at the same time, which corresponds to the case of one of the equal roots of the equation (*Art.* 180).

2°. If any term of the series, except the first and last, has a different sign in the row corresponding to the value p of the variable from that which it has in the row corresponding to the value q of the variable, it must vanish for some value of the variable contained between p and q. But for this value of the variable the preceding term must have a different sign from the succeeding term; thus, when

$$U''=0,$$

the equation $\qquad U'=k''U''-U'''$

gives $\qquad U'=-U'''.$

By the change of sign which the term undergoes in vanishing, therefore, it can only change from forming a permanence with one of its adjacent terms to forming one with the other of these terms, and *the change of sign of a term which is neither the first nor the last of the series does not increase or diminish the number of permanences of the row of the signs.*

3°. When the first term u of the series, in changing its sign, vanishes, while the second term U does not vanish, the corresponding value of the variable is a root of the equation which is not one of the equal roots. Also, if the variable is decreasing in value, the signs of these two terms constitute a permanence before the change and a variation after the change.

When the variable, therefore, in decreasing, passes through a value which is one of the unequal roots of the equation, the number of permanences in the row of signs is increased by unity.

4°. When the proposed equation has no equal roots, u and U can have no common divisor, and therefore the last term of the series will not contain the variable; it must, therefore, be of a constant value, and no change of sign can arise from it.

In this case, then, the number of permanences must, by the foregoing division of the proof, be greater in the row which corresponds to the greater of the two limits p *and* q, *than in the row which corresponds to the less of these two limits, by a number which is ex-*

actly equal to the number of real roots contained between p *and* q.

(189.) *If plus infinity is substituted for* p, *and negative infinity for* q *in the series of divisors, the resulting rows of signs show at once the whole number of real roots of the given equation.*

We shall now apply the theorem to a few

EXAMPLES.

1. Find the number of the roots of the equation

$$x^3-4x^2-6x+8=0.$$

Here we have $u = x^3-4x^2-6x+8$

$$U=3x^2-8x-6.$$

Find now the greatest common measure of u and U; thus, multiplying u by 3, to avoid fractions, it becomes

$$\begin{array}{r|l}
3x^3-12x^2-18x+24 & 3x^2-8x-6 \\ \cline{2-2}
3x^3-\ 8x^2-\ 6x\quad\quad & x-\frac{4}{3} \\ \cline{1-1}
-\ 4x^2-12x+24 & \\
-\ 4x^2+\frac{32x}{3}+\ 8 & \\ \cline{1-1}
-\frac{68x}{3}+16. &
\end{array}$$

Multiplying by 3, the remainder becomes

$$-68x+48;$$

$$\therefore\ U'-68x-48.$$

Multiplying $3x^2=8x-6$ by 68, it becomes

$$204x^2-544x-408$$

$$\begin{array}{r|l}
204x^2-544x-408 & 68x-48 \\ \cline{2-2}
204x^2-144x\quad\quad & 3x \\ \cline{1-1}
-400x-408 &
\end{array}$$

The next remainder will be minus.

Hence the series of functions are

$$
\begin{aligned}
u &= x^3-4x^2-6x+8 \\
U &= 3x^2-8x-6 \\
U' &= 68x-48 \\
U'' &= +.
\end{aligned}
$$

Substitute for x, in the leading terms of these functions, the values $+\infty$ and $-\infty$, and we shall have

For $x=+\infty$	$++++$	no variation;
$x=-\infty$	$-+-+$	three variations;

$\therefore 3-0=3=$the number of real roots in the equation.

To find the initial figures of the roots, we must employ narrower limits than $+\infty$ and $-\infty$. Beginning at zero, let us extend the limits both ways; and, since the proposed equation has only one permanence of sign, one of the roots is negative, and the remaining roots are positive.

For $x=0$,	we have	the signs	$+--+$	2	variations;
$x=1$	"	"	$--++$	1	"
$x=2$	"	"	$--++$	1	"
$x=3$	"	"	$--++$	1	"
$x=4$	"	"	$-+++$	1	"
$x=5$	"	"	$++++$	0	"
$x=6$	"	"	$++++$	0	"

We perceive, then, by the column of variations, that one of the roots is between 0 and 1, another between 4 and 5. Hence 0 and 4 are the initial figures of these two roots.

If we substitute

$x=0$,	we shall	have	the signs	$+--+$	2	variations;
$x=-1$	"	"	"	$++-+$	2	"
$x=-2$	"	"	"	$-+-+$	3	"

Hence the initial figure of the third root is -1, as it lies between -1 and -2.

If $x=1$, the signs are $--++$ one variation;
$x=.9$, " $+-++$ two variations.

Hence the initial figures of the roots are

$$-1,\ .9 \text{ and } 4.$$

2\. Find the roots of the equation

$$x^3+11x^2-102x+181=0.$$

The functions are,

$$\begin{aligned} u &= x^3+11x^2-102x+181 \\ U &=3x^2+22x-108 \\ U' &=122x-393 \\ U'' &=+. \end{aligned}$$

The signs of the leading terms are all *plus*, hence the substitution of $-\infty$ and $+\infty$ must give three real roots.

Making $x=0$ gives $+--+$ two variations;
$x=1$ " $+--+$ " "
$x=2$ " $+--+$ " "
$x=3$ " $+--+$ " "
$x=4$ " $++++$ no variation.

Hence the two positive roots are between 3 and 4.

3\. Find the number of real roots of the equation

$$x^4+x^3-x^2+2x+4=0.$$

Here the functions are,

$$\begin{aligned} u &= x^4+x^3-x^2-2x+4 \\ U &=4x^3+3x^2-2x-2 \\ U' &= x^2+2x-6 \\ U'' &=-x+1 \\ U''' &=+. \end{aligned}$$

Let $x=+\infty$, the signs are $+++-+$ 2 variations;
$x=-\infty$, " " $+-+++$ 2 "

Hence all the roots of the equation are imaginary.

4. Required the number of real roots of the equation

$$2x^4-11x^2+8x-16=0.$$

Ans. Two real roots, one positive, the other negative. The initial figures of the real roots are 2 and -2.

FIFTH METHOD OF ELIMINATION, CALLED ELIMINATION BY THE METHOD OF THE GREATEST COMMON DIVISOR.

(190.) If we transpose all the terms in each equation to the first member of the same, we shall have zero as the second member.

If, after having arranged the terms according to the powers of the quantity to be eliminated, we divide the first member of one equation by the first member of the second, the remainder arising from this division must be equal to zero; for this remainder is the difference between the dividend and a certain multiplier of the divisor, that is, between zero and a certain multiple of zero. Hence,

Divide one of these first members by the other, and proceed in the same manner as that to obtain the greatest common divisor; each successive remainder may be put equal to zero. But a remainder will at last be obtained which does not contain the quantity to be eliminated, and the equation formed from placing this remainder equal zero is the equation required.

If there are more than two equations, the same process is to be observed, until there is but one equation containing but one unknown quantity.

EXAMPLES.

1. Eliminate x from the equations

$$x^2+y^2-x-y-78=0 \quad . \quad . \quad . \quad . \quad (1)$$
$$xy+x+y-39=0 \quad . \quad . \quad . \quad . \quad (2)$$

The first equation can be written

$$x(x-1)+y(y-1)-78=0 \quad . \quad . \quad . \quad (3)$$

and the second $\quad x(y+1)+(y-39)=0 \quad . \quad . \quad . \quad (4)$

Multiply (3) by $(y+1)$, and we have

$$x(x-1)(y+1)+y(y^2-1)-78(y+1)=0.$$

Divide this first member by the first member of (4), and we shall have

$$\begin{array}{l|l} x(x-1)(y+1)+y(y^2-1)-78(y+1) & \underline{x(y+1)+(y-39)} \\ \underline{x(x-1)(y+1)+(x-1)(y-39)} & (x-1)=\text{quotient.} \\ \text{Rem.}= \; -(x-1)(y-39)+y(y^2-1)-78(y+1)=0. & \end{array}$$

Or, by actually multiplying $-(x+1)$ by $(y-39)$ and reducing, we have

$$-x(y-39)+(y-39)+y(y^2-1)-78(y+1)=0 \quad . \quad (5)$$

Multiplying (5) by $(y+1)$, it becomes

$$-x(y-39)(y+1)+(y-39)(y+1)+y(y^2-1)(y+1)$$
$$-78(y+1)^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Dividing (6) by (4), we shall have

$$\begin{array}{l|l} -x(y-39)(y+1)+(y-39)(y+1)+y(y^2-1)(y+1)-78(y+1)^2 & \\ \underline{-x(y-39)(y+1)-(y-39)^2} & \underline{x(y+1)+(y-39)}=\text{divisor.} \\ & -(y-39)=\text{quotient.} \\ \text{Rem.}= \; (y-39)^2+(y-39)(y+1)+y(y^2-1)(y+1)-78(y+1)^2=0. & \end{array}$$

Expanding and reducing, this becomes

$$y^4+y^3-77y^2-273y+1404=0,$$

the equation required.

2. Eliminate x from the equations

$$x^2+y^2=13,$$
$$x+y=5.$$

Ans. $y^2-5y+6=0.$

3. Eliminate x from the equations

$$x^3+x^2y+x+y=4,$$
$$x^3+x^2+xy=3.$$

Ans. $y-1=0$, or $y^2-3y+21=0.$

4. Find the value of x and y in the equations

$$x^2y-x^2+x=3,$$
$$xy(x^2y+1)-x^2+x=6.$$

The elimination of x gives $3y-3=0$, or $y=1$.

Ans. $x=3$.

(191.) *To transform an equation into another whose roots shall be the roots of the proposed equation increased or diminished by any given quantity.*

Let $ax^n+A_1x^{n-1}+A_2x^{n-2}+\ldots A_{n-1}x+A_n=0$ be the equation which it is required to transform into another whose roots shall be the roots of this equation diminished by r.

If we write $y+r$ for x in the given equation, it will be an equation of the same dimensions, and its form will evidently be

$$ay^n+B_1y^{n-1}+B_2y^{n-2}+\ldots B_{n-1}y+B_n=0 \quad . \ (1)$$

in which B_1, B_2, &c., will be polynomials involving r. But $y=x-r$, and therefore the above becomes

$$a(x-r)^n+B_1(x-r)^{n-1}+\ldots B_{n-1}(x-r)+B_n=0 \ . \ (2)$$

which, when developed, must be identical with the given equation. Hence

$$a(x-r)^n+B_1(x-r)^{n-2}+\ldots B_{n-1}(x-r)+B_n=ax^n+A_1x^{n-1}$$
$$+\ldots A_{n-1}x+A_n.$$

Now if we divide the first member by $x-r$, every term will evidently be divisible except the last B_n, which will be the remainder, and the quotient will be

$$a(x-r)^{n-1}+B_1(x-r)^{n-1}+\ldots B_{n-2}(x-r)+B_{n-1};$$

and since the second member is identical with the first, the same quotient and remainder would arise by dividing this second member also by $x-r$; hence it appears that if the first member of the original equation

be divided by $x-r$, the remainder will be the last or absolute term of the transformed equation required.

Again, if we divide the quotient thus obtained by $x-r$, the remainder will obviously be B_{n-1}, the coefficient of the term last but one in the transformed equation; and thus, by successive divisions of the polynomial in the first member of the given equation by $x-r$, we shall obtain the whole of the coefficients of the required equation.

The labor of dividing may be greatly reduced by *Horner's Synthetic Method* of division. Thus,

Ex. 1. Transform the equation

$$5x^4-12x^3+3x^2+4x-5=0$$

into another whose roots shall be less than those of the proposed equation by 2.

Here the coefficients are,

1st coef.	2d coef.	3d coef.	4th coef.	absolute term.		
5	-12	$+3$	$+4$	-5	(2	
	10	-4	-2	4		
	-2	-1	2	-1		$\therefore B_4=-1$
	10	16	30			
	8	15	32			$\therefore B_3=32$
	10	36				
	18	51				$\therefore B_2=51$
	10					
	28					$\therefore B_1=28;$

$$\therefore\ 5y^4+28y^3+51y^2+32y-1=0$$

is the transformed equation.

Note.—When there is a term absent in the equation, its place must be supplied with a cipher.

Ex. 2. Transform the equation

$$x^4-4x^3-8x+32=0$$

to another whose roots shall be less by 2.

This equation has no term containing x^2, therefore the coefficient of x^2 must be taken $=0$.

$$\begin{array}{rrrrrl}
1 & -4 & +0 & -\ 8 & +32 & (2 \\
 & 2 & -4 & -\ 8 & -32 & \\
\cline{2-5}
 & -2 & -4 & -16 & 0 & \therefore\ B_4=0 \\
 & 2 & -0 & -\ 8 & & \\
\cline{2-4}
 & 0 & -3 & -24 & & \therefore\ B_3=-24 \\
 & 2 & +4 & & & \\
\cline{2-3}
 & 2 & 0 & & & \therefore\ B_2=0 \\
 & 2 & & & & \\
\cline{2-2}
 & 4 & & & & \therefore\ B_1=4;
\end{array}$$

$$\therefore\ y^4+4y^3-24y=0$$

is the transformed equation.

Ex. 3. Transform the equation

$$x^4-3x^3-15x^2+49x-12=0$$

into another whose roots shall be less by 3.

Ans. $y^4+9y^3+12y^2-14y=0$.

(192.) HORNER'S METHOD OF RESOLVING NUMERICAL EQUATIONS OF ALL DEGREES.

RULE.

1. *Find the value of the first figure of the root by Sturm's Theorem.*

2. *Transform the equation into another whose roots shall be less than those of the proposed equation by the first figure already obtained.*

3. *Divide the absolute term of the transformed equation by the coefficient of the first power of the unknown quantity for the second figure of the root.*

4. *Transform the last equation into another whose roots shall be less, by the value of the last figure determined. Proceed in this manner until the whole root is determined, or until the approximation is carried sufficiently far.*

5. *To find negative roots, change the signs of the alternate terms, and proceed as for a positive root.*

The work may be considerably abridged in the following manner.

1. Let the absolute term constitute the second member of the equation, and take the algebraic difference of it and the number placed under it.

2. Omit writing out the transformed equations. There is a difficulty of obtaining the second figure of the root, but after this there is seldom any further uncertainty.

Examples.

1. Find a root of the equation

$$x^3-x^2+70x-300=0.$$

We find that one root lies between 3 and 4.

$$\begin{array}{rrrrl}
1 & -1 & +70 & =+300 & (3 \\
 & \underline{+3} & \underline{+\ 6} & \underline{+228} & \\
 & 2 & 76 & 72 & \\
 & 3 & \underline{15} & & \\
 & \overline{5} & 91 & & \\
 & 3 & & & \\
 & \overline{8} & & & \qquad \dfrac{72}{91}=.7.
\end{array}$$

$$
\begin{array}{rrrl}
1 & +\ 8 & +\ 91 & =+72\ (.7 \\
 & +\ .7 & +\ 6.09 & +67.963 \\ \cline{2-4}
 & 8.7 & 97.09 & \quad 4.037 \\
 & .7 & 6.58 & \\ \cline{2-3}
 & 9.4 & 103.67 & \\
 & .7 & & \\ \cline{2-2}
 & 10.1 & & \\ \hline
\end{array}
\qquad \frac{4.037}{103.67}=.03.
$$

$$
\begin{array}{rrrl}
1. & +10.1 & +103.67 & =+4.037\ (.03 \\
 & +\ .03 & +\ .3039 & +3.119217 \\ \cline{2-4}
 & 10.13 & 103.9739 & \quad .917783 \\
 & .03 & .3048 & \\ \cline{2-3}
 & 10.16 & 104.2787 & \\
 & 3 & & \\ \cline{2-2}
 & 10.19 & & \\
\end{array}
\qquad \frac{.917783}{104.2787}=.008.
$$

$\therefore$ One root of the equation is 3.738.

2. Find the roots of the equation

$$x^3+11x^2-102x+181=0.$$

We find that 3 is the first figure of one of the roots.

$$
\begin{array}{rrrl}
1 & +11 & -102 & =-181\ (3 \\
 & +\ 3 & +\ 42 & \ -180 \\ \cline{2-4}
 & 14 & -\ 60 & -\quad 1=\text{1st dividend.} \\
 & 3 & +\ 51 & \\ \cline{2-3}
 & 17 & -\ \ 9 & =\text{1st divisor.} \\
 & 3 & & \\ \cline{2-2}
 & 20 & & \\ \hline
\end{array}
\qquad \frac{1}{9}=.1.
$$

We shall find the second figure to be .2, and not .1.

$$
\begin{array}{rrrl}
1 & +20 & -9 & =-1\ (.2 \\
 & .2 & 4.04 & -.992 \\ \cline{2-4}
 & 20.2 & -4.96 & -.008=\text{2d dividend.} \\
 & .2 & 4.08 & \\ \cline{2-3}
 & 20.4 & -\ .88 & =\text{2d divisor.} \\
 & .2 & & \\ \cline{2-2}
 & 20.6 & & \\ \hline
\end{array}
\qquad \frac{.008}{.88}=.01.
$$

$$
\begin{array}{rrrl}
1 & +20.6 & -.88 & =-.008\,(.01 \\
 & .01 & +.2061 & \quad -.006739 \\
\hline
 & 20.61 & -.6739 & \quad .001261=\text{3d dividend.} \\
 & .01 & .2062 & \\
\hline
 & 20.62 & .4677 & =\text{3d divisor.} \\
 & .01 & & \\
\hline
 & 20.63 & &
\end{array}
\qquad \frac{.001261}{.4677}=.003.
$$

$\therefore$ One root is 3.213.

The other roots may be found in the same manner.

3. Find an approximate root of the equation

$$x^3+2x^2-23x-70=0.$$

Ans. $x=5.1345+$.

4. Find an approximate root of the equation

$$x^3+3x^2+5x-178=0.$$

Ans. $x=4.538$.

Generally, let $x^3+ax^2+bx=c$ be any equation of the third degree, r the first figure of one of the roots. Transform the equation into another whose roots shall be less by r; thus,

$$
\begin{array}{lll}
1+a & +b & =c\ (r \\
\underline{\quad r} & \underline{(a+r)r} & \underline{br+(a+r)r^2} \\
a+r & b+(a+r)r & c-br-(a+r)r^2 \\
\underline{\quad r} & \underline{(a+2r)r} & \\
a+2r & 2ar+3r^2+b & \\
\underline{\quad r} & & \\
a+3r & &
\end{array}
$$

The transformed equation is, therefore,

$$y^3+(a+3r)y^2+(2ar+3r^2+b)y=c-br-(a+r)r^2.$$

If we now put $(a+3)r=a'$, $(2ar+3r^2+b)=b'$, and $c-br-(a+r)r^2=c'$, we shall have

$$y^3+a'y^2+b'y=c',$$

an equation similar to the original equation.

If we transform this equation into another whose roots shall be less by r', we shall have

$$z^3+(a'+3r')z^2+(2a'r'+3r'^2+b')z=c'',$$

or $$z^3+a''z^2+b''z=c'',$$

an equation also similar to the original equation. Thus we may proceed, forming new equations all similar to the first, whose roots are constantly diminishing in value.

From the first transformation, we have

$$r=\frac{c}{b+(a+r)r} \text{ nearly.}$$

From the second,

$$r'=\frac{c'}{r'(r'+3r+a)+b'} \text{ nearly.}$$

From the third, when carried out,

$$r''=\frac{c''}{r''\{r''+3(r+r')+a\}+b''} \text{ nearly.}$$

The denominators of these fractions are complete divisors, and the quantities b', b'' are *trial* divisors. The further we proceed, the nearer will the trial divisor agree with the true divisors.

Find one of the roots of the equation

$$x^4-8x^3+14x^2+4x=8.$$

We find that one root lies between 5 and 6.

$$\begin{array}{rrrrl}
1 & -8 & +14 & +\ 4 & =\ \ 8\,(5=r \\
 & 5 & -15 & -\ 5 & =-5 \\
\hline
 & -3 & -\ 1 & -\ 1 & \ \ 13=c'=\text{1st dividend.} \\
 & 5 & 10 & 45 & \\
\hline
 & 2 & 9 & 44 & =\text{1st divisor.} \\
 & 5 & 35 & & \\
\hline
 & 7 & 44 & & \\
 & 5 & & & \\
\hline
 & 12 & & &
\end{array}$$

$$\frac{13}{44}=.2=r'.$$

$$
\begin{array}{lllll}
1+12 & +44 & +44 & =13\,(.2=r' & \\
\quad .2 & \quad 2.44 & \quad 9.288 & 10.6576 & \\
\cline{1-4}
\quad 12.2 & \quad 46.44 & \quad 53.288 & \quad 2.3424=c''=\text{2d dividend.} & \\
\quad .2 & \quad 2.48 & \quad 9.784 & & \\
\cline{1-3}
\quad 12.4 & \quad 48.92 & \quad 63.072=\text{2d divisor.} & & \\
\quad .2 & \quad 2.52 & & & \\
\cline{1-2}
\quad 12.6 & \quad 51.44 & & & \\
\quad .2 & & & & \\
\cline{1-1}
\quad 12.8 & & & \dfrac{2.3424}{63.072}=.03=r''. & \\
\hline
1+12.8 & +51.44 & +63.072 & =2.3424\,(.03=r'' & \\
\quad .03 & \quad .3849 & \quad 1.554747 & 1.93880241 & \\
\cline{1-4}
\quad 12.83 & \quad 51.8249 & \quad 64.626747 & \quad .40359759=c'''=\text{3d dividend.} & \\
\quad .03 & \quad .3858 & \quad 1.566321 & & \\
\cline{1-3}
\quad 12.86 & \quad 52.2107 & \quad 66.193068=\text{3d divisor.} & & \\
\end{array}
$$

$$\therefore\ x=5.236. \qquad \frac{.40359759}{66.193068}=.006=r'''.$$

The work may be considerably abridged by performing the additions at once, and continuing the process without bringing down the transformations. It would stand thus,

$$
\begin{array}{llll}
1-8 & +14 & +4 & =8\,(5.236 \\
\ -3 & -\ 1 & -1 & -5 \\
\cline{4-4}
\ +2 & +\ 9 & +44 & \ 13 \\
\quad 7 & \quad 44 & \quad 53.288 & 10.6576 \\
\cline{4-4}
\quad 12.2 & \quad 46.44 & \quad 63.072 & \ 2.3424 \\
\quad 12.4 & \quad 48.92 & \quad 64.626747 & \ 1.93880241 \\
\cline{4-4}
\quad 12.6 & \quad 51.44 & \quad 66.193068 & \ .40359759 \\
\quad 12.83 & \quad 51.8249 & \quad 66.509117736 & \\
\quad 12.86 & & & \\
\end{array}
$$

QUESTIONS FOR ORAL EXAMINATION.

What is Algebra? How do you explain the signs *plus* and *minus*? What is a coefficient? What is an exponent? What is the power of a quantity? What is the root of a quantity? What is the index of the root? What is the reciprocal of a quantity? What is a monomial? What, a polynomial? What, a binomial? What, a trinomial? What is the numerical value of a polynomial? What are additive terms? What, subtractive? What does each literal factor constitute? What constitutes a degree? How is the degree determined? When is a polynomial homogeneous? What is the use of the parenthesis? When is the vinculum placed under the polynomial? What are similar terms? What is the rule for reduction of similar terms? What is Addition? What is the rule for Addition? How may quantities with literal coefficients be added? What is Subtraction? Give the rule. When the sign before a polynomial inclosed in a parenthesis is minus, and the parenthesis is taken away, what is to be done? Do addition and subtraction in Algebra always mean increase and diminution? Give an example in each. What is the rule for multiplication of monomials? What, for the multiplication of a polynomial by a monomial? What is the rule for the multiplication of polynomials? If the factors are homogeneous, what will their product be? What the degree of each term? What is the square of the sum of two quantities? What is the square of the difference of two quantities? What is the difference of the squares of two quantities equal to? What is Division? What is the rule for the division of monomials? What is a to the exponent zero equal to? How may a factor be taken from the numerator into the denominator? Rule for the division of polynomials. How may a polynomial sometimes be factored? The difference of the same powers of any two quantities is always divisible by what? The difference of two even powers of the same degree is divisible by what? The sum of two odd powers is divisible by what? What is the form of an even number? What, of an odd number? What is said of algebraic fractions? What are the three principles in regard to fractions? How do you reduce a fraction to its simplest form? What is the greatest common measure of two or more quantities? How do you find the

greatest common measure? How do you reduce a mixed quantity to the form of a fraction? What is a mixed quantity? How do you reduce a fraction to an entire or mixed quantity? How do you find the least common multiple of two or more quantities? How do you reduce fractions to the least common denominator? How do you add fractional quantities? How do you subtract one fractional quantity from another? Give the rule for multiplication of fractions. For division.

What is an equation? Of what does it consist? What is the difference between a member and a term? What is an identical equation? When is an equation verified? How are equations divided? What is an equation of the first degree? What is a numerical equation? What, a literal equation? On what are founded the different transformations of equations? How is a term transposed? When a fractional term is preceded by the sign *minus*, and the fraction is made entire, what is done with the numerator? How is an equation cleared of fractions? Give the rule for solving an equation of the first degree containing but one unknown quantity. How may a proportion be converted into an equation? What does every problem include in its enunciation? How do you form an equation? What is elimination? How many different methods of elimination? What are they? What is the rule for elimination by addition or subtraction? By substitution? By comparison? By an indeterminate multiplier? Suppose we have m equations containing m unknown quantities, how will you eliminate them? What is the usual method of solving problems of the first degree involving more than one unknown quantity? What does a negative value show? Discuss the problem of the two pedestrians. If m is less than n, what is the value of t? If m is greater than n, what is the value of t? Explain it. If m is equal to n, what is the value of t? Interpret it. If a is equal to zero and $m=n$, interpret it. Is zero divided by zero always indeterminate? Give an example.

What is an inequality? Repeat the different transformations. What is the square root of a number? How may every number be regarded? The square of a number composed of tens and units consists of what? Rule for extracting the square root of numbers. The difference of the squares of two consecutive numbers is equal to what? How do you extract the square root of a number that is not a perfect square to within a given fraction?

What are the powers of a quantity? How do you square a monomial? How do you raise a monomial to any power? A polynomial? What is the square of any polynomial?

What is the square root of an algebraic quantity? How do you extract the square root of a monomial? The root of a fraction? When is a monomial a perfect square? What is the even root of a negative quantity? What is a radical of the second degree? The product of the square roots of any number of factors is equal to what? What is the coefficient of a radical? How do you simplify a radical of the second degree? What are similar radicals? How do you add or subtract similar radicals? What is the product of two radicals of the second degree equal to? What is the quotient? How do you pass the coefficient of a radical under the radical sign? What is the object of this transformation? How is the binomial denominator of a surd fraction made rational? Give the rule for extracting the square root of a polynomial. How do you raise a radical to any power? How do you reduce radicals to a common index? When is a polynomial an imperfect square? When is a binomial a perfect square? When, a trinomial? How do you extract the root of a trinomial which is a perfect square? How do you simplify a polynomial that is not a perfect square? Give the rule for extracting the *n*th root of a polynomial.

What is an equation of the first degree? Of the second degree? How are equations of the second degree divided? What is a complete equation of the second degree? An incomplete? What is the form of an incomplete equation? What is it sometimes called? What is said of the roots of an incomplete equation? Give the rule for the solution of an incomplete equation. When the equation is reduced to the form $x^{\frac{m}{n}}=a$. What is the form of a complete equation of the second degree? How do you solve a complete equation of the second degree? What is the first value of the unknown quantity? What the second? To what is the rule just given equally applicable? How many roots does an equation have?

What is an arithmetical progression? What is the last term of an increasing series? Decreasing series? What is the sum of the terms? What is a geometrical progression? What is the constant multiplier called? When is the series an increasing one? When a decreasing one? How do you find the sum of an increasing geometrical series? The sum of any geometrical series? The sum of an infinite number of terms of a decreasing series? How do you find any number of geometrical means between two numbers? How do you find the ratio? What are permutations? How do you find the number of permutations of n quantities taken two and two? Taken three and three? Taken m and m together? What are combina-

tions? How do you find the number of combinations of n quantities taken m and m together? What are the coefficients of a binomial? What is the sum of the coefficients of the nth power of a binomial equal to? What is the relation of the coefficients of the two middle terms? Of those equally distant from the two extremes? What are the terms without the coefficients? How is the coefficient of any term found? On what does the principle of indeterminate coefficients consist? Give the rule for converting an algebraic expression into a series, by means of indeterminate coefficients. What is the formula for the summation of an infinite series?

What is a continued fraction? What are approximating fractions? Why are they so called? How do you find the numerator of the nth approximating fraction? Denominator? How do you convert an irreducible fraction into a continued fraction? What are logarithms? The logarithm of any number? How many systems of logarithms may there be for the same number? What is the logarithm of 1 in any system? What is the logarithm of the product of two or more factors? How is the mth power of a number obtained by logarithms? What is the logarithm of a fraction? How is the nth root of a number obtained by logarithms? The series for the logarithm of any number is composed of how many factors? Which is called the modulus? How are the logarithms of the same number taken in two different systems to each other? If you have the Naperian logarithm of a number, how would you find the logarithm of the same number in another system? The common logarithm divided by the modulus will give what? What are exponential equations?

What is a function? What is the derivative of a function? The derivative of the sum of two functions is equal to what? The derivative of the difference of two functions is equal to what? How do you find the derivative of any power of a variable? Of a function? Of the product of two functions? What is the general form of an equation of nth degree containing but one unknown quantity? By what is the number of roots of an equation denoted? What is the coefficient of x^2 in the general equation of the third degree? Of x? The term which does not contain x? Apply this to the equation of the nth degree. How do you find the equal roots of an equation? If the equation has no real root? When an uneven number of the real roots of an equation are comprehended between two numbers, what then? If an even number? What is said of equations of an uneven degree? Of an even degree? The number of real positive roots? What constitutes a permanence? What a variation? How is the number of real roots found by Sturm's theorem? What two

characters, being substituted in the series of functions, will show at once the number of real roots? What is the fifth method of elimination? Give the rule. Give the rule for Horner's method of solving numerical equations of any degree.

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